On the regularity of weak solutions to refractor problem

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Abstract

In this note we derive the Monge-Ampère type equation in Euclidian coordinates describing the refraction phenomena of perfect lens. This simplifies the regularity issues of the weak solutions on the problem.

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1 Introduction and main result

It is well-known that ellipse and hyperbola have simple refraction properties, namely if rays of light diverge from one focus, then after refraction they pass parallel to the major axis [4]. If the ellipse (resp. hyperbola) represents the boundary separating two medias, with refractive indices \( n_1, n_2 \) then according to refraction law

\[
    n_1 \sin \alpha = n_2 \sin \beta,
\]

where \( \alpha \) and \( \beta \) are the angles between normal and respectively the ray before and after refraction. Let \( k = n_1/n_2 \), then one can verify that \( k = 1/\varepsilon \), where \( \varepsilon \) is the eccentricity of ellipse (resp. hyperbola) [4]. These properties are limiting cases of solutions to more general problems of determining the surface required to refract rays of light diverging from one point and after refraction covering a given set of directions on the unit sphere. More precisely let us assume we are given two sets \( \Omega, \Omega^* \) on unit sphere centered at origin, and nonnegative integrable functions \( f, g \) defined respectively on \( \Omega \) and \( \Omega^* \). Suppose that a point source of light is centered at the origin \( O \) and for every \( X \in \Omega \) we issue a ray from \( O \) passing through \( X \), which after refraction from the unknown surface \( \Gamma \) is another ray given by a unit direction...
\[ Y = Y(X) \in \Omega^* \]. It is clear that mapping \( Y \) is determined by \( \Gamma \). Let \( f(X) \) be the input intensity of light at \( X \in \Omega \) and \( g(Y) \) corresponding gain intensity after refraction at \( Y \in \Omega^* \). Now the problem can be formulated as follows: given two pairs \( (\Omega, f) \) and \( (\Omega^*, g) \) satisfying to energy balance condition

\[
\int_{\Omega} f = \int_{\Omega^*} g, \tag{1.1}
\]

find a surface \( \Gamma \), such that for corresponding mapping \( Y(X) \) we have

\[
Y(\Omega) = \Omega^*.
\]

We seek a \( \Gamma \) as a radial graph of a unknown function \( \rho \) i.e. \( \Gamma = \{ Z \in \mathbb{R}^{n+1}, Z = X \rho(X) \} \), then mathematically this problem is amount to solve a Monge-Ampère type equation

\[
\det(D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho)) = h(x, \rho, D\rho), \tag{1.2}
\]

subject to boundary condition

\[
Y(\Omega) = \Omega^*. \tag{1.3}
\]

Here the derivatives are taken in some orthogonal coordinate system (see Theorem 1) and \( \Omega \) is a subset of upper half sphere. The solutions to (1.2), should be sought in the class of functions such that the matrix \( D^2_{ij}\rho - \sigma_{ij}(x, \rho, D\rho) \geq 0 \). It is easy to see that if \( \rho \in C^2 \) such that \( D^2\rho - \sigma_{ij} \geq 0 \) then equation (1.2) is elliptic with respect to \( \rho \).

It turns out that \( \rho \) is a potential function to an optimal transfer problem with a logarithmic cost function [1]

\[
c(X, Y) = \begin{cases} 
\log \frac{1}{1-\epsilon(Y-X)}, & \epsilon > 1, \ X \cdot Y > k, \\
\log \frac{1}{1+\epsilon(Y-X)}, & \epsilon < 1, \ X \cdot Y < k.
\end{cases}
\]

A similar cost function appears in the reflector problem introduced by X-J. Wang [8], [9]. The regularity of the solutions to optimal transfer problems is discussed in [3] and [5]. The most important thing is the so-called A3 condition, imposed on matrix \( \sigma_{ij} \) [3]. As soon as one has it the rest of the regularity, both local and global will follow from the classical framework established in [3], [2] and [6]. In [11] authors have verified the A3 condition, however without using Euclidian coordinates.

In this note we give a simple way of verifying the A3 condition, for \( k < 1 \) without invoking to covariant derivatives. It is also explicit, strict and straightforward (3.4). Main idea is to find a simple formula for mapping \( Y(X) \) using a parametrization of upper unit half sphere, used in [2]. Then the rest will follow along the arguments of [2]. This method is very general and one can apply it to near-field problem. Indeed if one considers a map \( z = \rho x + ty, \) where \( t \) is the stretch function, then \( \det Dz \) will give the equation for near-field problem. However we don’t discuss this problem in the present note. It is worth noting that, if support functions are ellipsoids, i.e. \( k > 1 \) the A3 condition is not fulfilled (see (3.4)).
1.1 Notations

Let us consider the case of two homogeneous medias, with refractive constants $n_1$ and $n_2$. $\Omega$ and $\Omega^*$ are two domains on the unit sphere $S^n = \{X = (x_1, \ldots, x_{n+1}), x_1^2 + \cdots + x_{n+1}^2 = 1\}$. For $X \in S^n, x = (x_1, \ldots, x_n, 0)$. We also suppose that $\Omega$ is a subset of upper unit sphere $S^+_n = S^n \cap \{x_{n+1} > 0\}$. In what follows we consider $\rho$ as a function of $x \in \Omega_0$, with $\Omega_0$ as orthogonal projection of $\Omega$ on to hyperplane $x_{n+1} = 0$. By $D\rho$ we denote the gradient of function $\rho$ with respect to $x$ variable $D\rho = (Dx_1\rho, \ldots, Dx_n\rho, 0)$. The reciprocal of $\rho$ is defined as $u = 1/\rho$. We also define two auxiliary functions $b = u^2 + |Du|^2 - (Du \cdot x)^2$ and $V = \sqrt{u^2 - \sigma b} + u$. In what follows $\sigma = (k^2 - 1)/k^2 = 1 - \varepsilon^2$.

1.2 The main results

Our main result is contained in the following

**Theorem 1** If $\rho$ is the radial function defining $\Gamma$, and $u = 1/\rho$, then $u$ is a weak solution to

$$
\begin{align*}
\det \left\{ \frac{V - \sigma(u - Du \cdot x)}{\sigma} \left( Id + \frac{x \otimes x}{1 - |x|^2} \right) - D^2u \right\} &= h, \text{ if } k < 1, \\
\det \left\{ D^2u - \frac{V - \sigma(u - Du \cdot x)}{\sigma} \left( Id + \frac{x \otimes x}{1 - |x|^2} \right) \right\} &= h, \text{ if } k > 1,
\end{align*}
$$

where $b = u^2 + |Du|^2 - (Du \cdot x)^2$, $V = \sqrt{u^2 - \sigma b} + u$.

If we set $F = \sigma^{-1}(V - \sigma(u - Du \cdot x))$ and $I = Id + \frac{x \otimes x}{1 - |x|^2}$, the first fundamental form of the upper unit half sphere, then equation can be rewritten as $\det(IF - D^2u) = h$ for $k < 1$. The weak solutions for this equations can be defined through the theory of optimal transfers [1] (see [7] for the discussion of such problems). The higher regularity of the weak solutions depends on the properties of the function $F$. More precisely we have

**Theorem 2** If $k < 1$ (i.e. when the support functions are hyperboloids of revolution touching $\Gamma$ from below) and $(f, \Omega)$ and $(g, \Omega^*)$ satisfy to the regularity assumptions as in [3], [5] and [6] then $F$ is strictly concave as a function of the gradient and the weak solutions are locally (globally) smooth provided $f, g$ are positive smooth functions and $\Omega, \Omega^* \subset S^+_n$.

If $k > 1$ (i.e. when the support functions are ellipsoids of revolution touching $\Gamma$ from above) then $F$ is not convex in gradient and the weak solutions may not be $C^1$ even for smooth positive intensities $f, g$.

2 The main formulas

In this section we derive a simple and useful formula for $Y$. We use it to compute the Jacobian determinant in the next section.
2.1 The mapping $Y$

Let $Y$ be the unit direction of the refracted ray. First let us derive a formula for $Y$, using angles $\alpha$ and $\beta$ (see figure 1). Since $X, Y$ and outward unit vector $\gamma$ lie in the same plane, we have

$$Y = C_1X + C_2\gamma$$

for two unknowns, $C_1$ and $C_2$ depending on $X$. If one takes the scalar product of $Y$ with $\gamma$ and then with $X$, then

$$\begin{cases} 
\cos \beta = C_1 \cos \alpha + C_2 \\
\cos (\alpha - \beta) = C_1 + C_2 \cos \alpha.
\end{cases}$$

Multiplying the first equation by $\cos \alpha$ and subtracting from the second one we infer

$$C_1 = \frac{\sin \beta}{\sin \alpha}, \quad C_2 = \cos \beta - C_1 \cos \alpha.$$

Introduce $k = n_1/n_2$, hence we find that $C_1 = k$ and $C_2 = \cos \beta - k \cos \alpha$, that is

$$Y = kX + (\cos \beta - k \cos \alpha)\gamma. \quad (2.1)$$

We can further manipulate (2.1). Note that

$$n_2^2 - n_2^2 \cos^2 \beta = n_2^2 \sin^2 \beta = n_1^2 \sin^2 \alpha = n_1^2 - n_1^2 \cos^2 \alpha.$$

Dividing the both sides by $n_2^2$ we obtain

$$k^2 \cos^2 \alpha = (k^2 - 1) + \cos^2 \beta.$$
Returning to (2.1) we get
\[ Y = kX + \left( \sqrt{k^2 \cos^2 \alpha - (k^2 - 1)} - k \cos \alpha \right) \gamma = \] (2.2)
\[ = k \left( X + \sqrt{(X \cdot \gamma)^2 - \sigma - X \cdot \gamma} \right), \]
where \( \sigma = (k^2 - 1)/k^2 \). From [2] we have
\[ \gamma = -\frac{D\rho - X(\rho + D\rho \cdot x)}{\sqrt{\rho^2 + |D\rho|^2 -(D\rho \cdot x)^2}} \]
where \( X = (x, \sqrt{1 - |x|^2}), D\rho = (\rho_1, \ldots, \rho_n) \). It is convenient to work with a new function \( u = \rho^{-1} \). By direct computation we have that
\[ \gamma = \frac{Du + X(u - Du \cdot x)}{\sqrt{u^2 + |Du|^2 -(Du \cdot x)^2}}. \]
Introduce \( b = u^2 + |Du|^2 -(Du \cdot x)^2 \), then
\[ Y = k \left( X + \sqrt{(X \cdot \gamma)^2 - \sigma - X \cdot \gamma} \right) \]
\[ = k \left( X + \sqrt{\frac{u^2}{b} - \sigma - \frac{u}{\sqrt{b}} \gamma} \right) \]
\[ = k \left( X + b^{-1}\sqrt{u^2 - \sigma b - u}[Du + X(u - Du \cdot x)] \right), \]
where we used the fact that
\[ X \cdot \gamma = \frac{u}{\sqrt{u^2 + |Du|^2 -(Du \cdot x)^2}} > 0. \]
In particular it follows from the previous formula that
\[ Y_{n+1} = kX_{n+1}(1 - \frac{\sigma}{V}(u - (Du \cdot x))). \] (2.3)

2.2 The Jacobian determinant

Let \( dX \) and \( dY \) be respectively the area elements corresponding to \( \Omega \) and \( \Omega^* \). Then \( dx = X_{n+1}dX \). Recall that \( Y \) is a unit vector and denote \( y = (Y_1, Y_2, \ldots, Y_n, 0) \in \Omega^*_0 \), where \( \Omega^*_0 \) is the orthogonal projection of \( \Omega^* \) onto hyperplane \( x_{n+1} = 0 \) so we conclude \( dy = Y_{n+1}dY \). Hence if we consider \( y \) to be a mapping from \( \Omega_0 \) to \( \Omega^*_0 \) then \( dy = |\det Dy|dx \).

For perfect refractor \( \Gamma \) we have the energy balance condition
\[ \int_E f(X)dX = \int_{Y(E)} g(Y)dY, \forall \text{ measurable } E \subset \Omega. \]
Thus we obtain \( f dX = g dY \) or
\[ J = \frac{X_{n+1}}{Y_{n+1}} |\det Dy| = \frac{f(X)}{g(Y)} = \frac{dX}{dY}. \] (2.4)
Thus to find the Jacobian determinant $J$ it is enough to compute $|\det Dy|$. Before starting our computations let us note, that if $\mu = \text{Id} + C\xi \otimes \eta$ for some constant $C$ and for any two vectors $\xi, \eta \in \mathbb{R}^n$, then one has

$$\mu^{-1} = \text{Id} - \frac{C\xi \otimes \eta}{1 + C(\xi \cdot \eta)}, \quad \det \mu = 1 + C(\xi \cdot \eta). \quad (2.5)$$

### 3 Proofs of Theorems 1-2

The main goal of this section is to prove the following

**Proposition 1** If $Y$ is given as above and

$$y = k \left[ x - \sigma \frac{V}{V} (Du + x(u - Du \cdot x)) \right],$$

then

$$Dy = k\sigma \mu \left[ \text{Id} - x \otimes x \right] \left\{ (\text{Id} + \frac{x \otimes x}{1 - |x|^2}) \frac{V - \sigma (u - Du \cdot x)}{\sigma} - D^2 u \right\},$$

where $b = u^2 + |Du|^2 - (Du \cdot x)^2, V = \sqrt{u^2 - \sigma b + u}$ and $\mu$ is defined by (3.2).

**Proof.** Introduce $V = \sqrt{u^2 - \sigma b + u}, z = Du + x(u - Du \cdot x)$. Using these notations one can rewrite

$$y = k[x - \sigma \frac{V}{V} z].$$

By a direct computation we have

$$\frac{y_{ij}}{k} = \delta_{ij} - \frac{\delta}{V} (z_{j} - \frac{z^i V_j}{V}).$$

Differentiating $z^i$ and $V$ with respect $x_j$ yields

$$z_{j}^i = u_{ij} - x_i x_m u_{m,j} + \delta_{ij}(u - Du \cdot x),$$

$$V_j = pu_{ij} - q(u_m - (Du \cdot x)x_m)u_{mj},$$

where

$$p = \frac{V - \sigma(u - Du \cdot x)}{V - u},$$
$$q = \frac{\sigma}{V - u}.$$
Then
\[
\frac{Dy}{k} = \text{Id} - \sigma V \left[ (\text{Id} - x \otimes x)D^2u + \text{Id}(u - Du \cdot x) - \frac{p}{V} z \otimes Du \right]
+ \frac{q}{V} z \otimes (Du - (Du \cdot x)D^2u) \tag{3.1}
\]
\[
= \left[ 1 - \frac{\sigma}{V} (u - Du \cdot x) \right] \left[ \text{Id} + Az \otimes Du \\
- B \left\{ (Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)) \right\} D^2u \right],
\]
where we set
\[
A = \frac{\sigma p}{Vz} - \frac{\sigma}{V(u - Du \cdot x)} = \frac{\sigma}{V(V - u)},
\]
\[
B = \frac{\sigma}{1 - \frac{\sigma}{V}(u - Du \cdot x)} = \frac{\sigma}{V - \sigma(u - Du \cdot x)}.
\]

Then using Lemma 1 (see below) we finally obtain
\[
\frac{Dy}{k} = \left[ 1 - \frac{\sigma}{V} (u - Du \cdot x) \right] B\mu \left[ Id - x \otimes x \right] \left\{ (Id + \frac{x \otimes x}{1 - |x|^2}) \frac{1}{B} - D^2u \right\}
= \frac{\sigma}{V} \mu [Id - x \otimes x] \left\{ (Id + \frac{x \otimes x}{1 - |x|^2}) \frac{1}{B} - D^2u \right\}
= \frac{\sigma}{V} \mu [Id - x \otimes x] \left\{ \frac{V - \sigma(u - Du \cdot x)}{\sigma} (Id + \frac{x \otimes x}{1 - |x|^2}) - D^2u \right\}.
\]

Hence to finish the proof of Proposition 1 it remains to prove

**Lemma 1** Let \( \mu = \text{Id} + Az \otimes Du \), then
\[
\mu^{-1} \left\{ (Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x) \right\} = Id - x \otimes x, \tag{3.2}
\]
\[
det \mu = \frac{Y_{n+1}}{kX_{n+1}} \frac{u}{\sqrt{u^2 - \sigma b}}. \tag{3.3}
\]

**Proof.** First by (2.5)
\[
\mu^{-1} = \text{Id} - \frac{Az \otimes Du}{1 + A(z \cdot Du)}.
\]

Let \( \mathcal{N} = \left\{ (Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x) \right\} \), then by a direct computation we have
\[
\mu^{-1} \mathcal{N} = (Id - x \otimes x) + \frac{q}{V} z \otimes (Du - (Du \cdot x)x) - \frac{Az \otimes Du}{1 + A(z \cdot Du)}
+ \frac{A}{1 + A(z \cdot Du)}[(Du \cdot x)z \otimes x - \frac{q}{V}(Du \cdot z)z \otimes (Du - (Du \cdot x)x)].
\]
Let us sum up all $\otimes$ products with $z$, the resulting vector is

$$
\frac{q}{V}(Du - (Du \cdot x)x) + \frac{A}{1 + A(z \cdot Du)} \left\{ - Du + (Du \cdot x)x - \frac{q}{V}(Du \cdot z)(Du - (Du \cdot x)x) \right\} = \left\{ \frac{q}{V} - \frac{A}{1 + A(z \cdot Du)} \left( 1 + \frac{q}{V} Du \cdot z \right) \right\} (Du - (Du \cdot x)x).
$$

On the other hand

$$
\frac{q}{V} - \frac{A}{1 + A(z \cdot Du)} (1 + \frac{q}{V} Du \cdot z) = \frac{1}{1 + A(z \cdot Du)} [\frac{q}{V} - A].
$$

Using definitions of $q, p$ and $A$ we obtain that

$$
\frac{q}{V} - A = \frac{\sigma}{V(V - u)} - \frac{\sigma p}{V - \sigma(u - Du \cdot x)} = \frac{\sigma}{V} \left\{ \frac{1}{V - u} - \frac{V - \sigma(u - Du \cdot x)}{V - u} \right\} = 0.
$$

To prove (3.3) we notice that $A = \frac{\sigma}{V(V - u)}$. Then using (2.5) and $V = \sqrt{u^2 - \sigma b + u}$ we have

$$
det \mu = 1 + \frac{\sigma}{V(V - u)} [Du]^2 + u(Du \cdot x) - (Du \cdot x)^2
$$

$$
= \frac{1}{V(V - u)} [uV - u^2 \sigma + \sigma u(Du \cdot x)]
$$

$$
= \frac{u}{V - u} \left\{ 1 - \frac{\sigma}{V}(u - (Du \cdot x)) \right\}
$$

and (3.3) follows from (2.3).

3.1 Ellipsoids and hyperboloids of revolution

In this section we show that $W = IF - D^2 u \equiv 0$ for $u = \frac{1}{C}(1 - \varepsilon(\ell \cdot X))$, that is when $\rho = 1/u$ is the radial graph of ellipsoid or hyperboloid of revolution. To fix ideas we assume that $\ell = e_{n+1}$. Thus $u = \frac{1}{C}(1 - \varepsilon X_{n+1})$. It is enough to show that $B = CX_{n+1}/\varepsilon$. By direct computation

$$
b = \frac{1}{C^2}(1 - 2\varepsilon X_{n+1} + \varepsilon^2)
$$

$$
u^2 - \sigma b = \frac{\varepsilon^2}{C^2} (X_{n+1}^2 - \varepsilon^2).
$$

Therefore $V = (1 - \varepsilon^2)/C$, which implies that

$$
B = \frac{\sigma}{V - \sigma(u - Du \cdot x)} = \frac{CX_{n+1}}{\varepsilon}.
$$
3.2 Proof of Theorem 1

From Proposition 1, Lemma 1 and (2.5) we have that

\[
\det D_y = \left( k^{\sigma} \right)^n \det \mu \det (Id - x \otimes x) \det W
= \left( k^{\sigma} \right)^n \frac{Y_{n+1}}{k X_{n+1}} \frac{u}{\sqrt{u^2 - \sigma b}} (1 - |x|^2) \det W
\]

where \( W = IF - D^2 u \) i.e.

\[
W = \frac{V - \sigma (u - Du \cdot x)}{\sigma} (Id + \frac{x \otimes x}{1 - |x|^2}) - D^2 u.
\]

Notice that if \( \Gamma \) is smooth and has support hyperboloids from inside at each point then \( W \geq 0 \) and \( W \leq 0 \) if \( \Gamma \) has support ellipsoids from outside. Then from (2.4) the Theorem 1 follows.

3.3 Proof of Theorem 2

The equation (1.4) is generalized Monge-Ampère equation. To obtain smoothness of the solution, one needs to show, that \( F = \frac{V - \sigma (u - Du \cdot x)}{\sigma} \) is strictly concave in gradient. This is a necessary condition, called A3 and first introduced in [3], in order to obtain \( C^2 \) a priori estimates. It turns out that if \( \sigma < 0 \), i.e. when support functions are hyperboloids of revolution, then \( F \) is strictly concave in gradient. Recall that \( V = \sqrt{u^2 - \sigma b} + u \), hence it is enough to show that \( \sqrt{u^2 - \sigma b} \) is convex in gradient. Let \( \xi \) be the dummy variable for \( Du \), then we have

\[
\frac{\partial}{\partial \xi_k} \sqrt{u^2 - \sigma b} = -\frac{\sigma}{\sqrt{u^2 - \sigma b}} (\xi_k - (\xi \cdot x)x_k),
\]

\[
\frac{\partial^2}{\partial \xi_k \partial \xi_l} \sqrt{u^2 - \sigma b} = -\frac{\sigma}{\sqrt{u^2 - \sigma b}} \left\{ \delta_{lk} - x_k x_l + \frac{(\xi_k - (\xi \cdot x)x_k)(\xi_l - (\xi \cdot x)x_l)}{u^2 - \sigma b} \right\}.
\]

On the other hand \( b = u^2 + |\xi|^2 - (\xi \cdot x)^2 \), which is strictly convex function of \( \xi \), provided \( |x| < 1 \). For any \( \eta \in \mathbb{R}^n \) we have

\[
-(u^2 - \sigma b)^2 \frac{\partial^2 F}{\partial \xi_k \partial \xi_l} \eta_k \eta_l = -(u^2 - \sigma b)^2 \frac{1}{\sigma} \sum_{k,l} \frac{\partial^2 \sqrt{u^2 - \sigma b}}{\partial \xi_k \partial \xi_l} \eta_k \eta_l
\]

\[
= (u^2 - \sigma b)(|\eta|^2 - (\eta \cdot x)^2) + \sigma |\xi \cdot \eta - (\xi \cdot \eta)(\eta \cdot \xi)|^2
\]

substituting the value of \( b = |\xi|^2 - (\xi \cdot x)^2 + u^2 \) we have

\[
= u^2 (1 - \sigma)(|\eta|^2 - (\eta \cdot x)^2) + \sigma \left\{ -|\xi|^2 |\eta|^2 + |\xi|^2 (\eta \cdot x)^2 + |\eta|^2 (\xi \cdot x)^2 + (\xi \cdot \eta)^2 - 2(\xi \cdot \eta)(\xi \cdot x)(\eta \cdot x) \right\}.
\]

The first term is nonnegative since \( |x| < 1 \) and \( (\eta \cdot x) \leq |\eta| |x| < |\eta| \). Recall that \( \sigma < 0 \). Hence it is enough to show that

\[
-|\xi|^2 |\eta|^2 + |\xi|^2 (\eta \cdot x)^2 + |\eta|^2 (\xi \cdot x)^2 + (\xi \cdot \eta)^2 - 2(\xi \cdot \eta)(\xi \cdot x)(\eta \cdot x) < 0.
\]

(3.5)
This expression is homogeneous in $\eta$ and $\xi$ thus we may assume that $|\xi| = |\eta| = 1$. Furthermore let $x'$ be the orthogonal projection of $x$ on the two dimensional space spanned by $\xi$ and $\eta$. Then (3.5) is equivalent to

$$(\eta \cdot x')^2 + (\xi \cdot x')^2 + (\xi \cdot \eta)^2 - 2(\xi \cdot \eta)(\xi \cdot x')(\eta \cdot x') - 1 < 0.$$  

If $\alpha, \beta$ and $\gamma$ are the angles between respectively $\eta$ and $x'$, $\xi$ and $x'$ and $\eta$ and $\xi$ then $\cos \gamma = \cos(\alpha \pm \beta)$. Thus we have

$$|x'|^2(\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \gamma) + \cos^2 \gamma - 1 <$$

$$+ (\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \gamma) + \cos^2 \gamma - 1 = 0.$$  

From here the proof of Theorem 1 follows from [3] and [6].

References


