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On Characteristic Function of a Contraction, Its Model and Function of Strauss

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Abstract. It is shown that the Nagy-Foias characteristic function of a completely nonunitary contraction and a variant of its functional model can be represented by means of a projection-valued analytic operator function, arising in the representation theory of A.V. Strauss.

Key Words: characteristic function of a contraction, functional model, projection-valued function of Strauss, reproducing kernel.
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Introduction

The present note is rather of a methodical character. A completely nonunitary contraction in a Hilbert space is discussed. Motivated by the work of Sh.N. Saakyan [2], the approach proposed here to a characteristic function and a functional model of a such contraction is based on its partial isometric dilation, acting in a doubled Hilbert space, that somewhat differs from considerations in [2].

Directly related to a characteristic function, a maximal function of a contraction introduced by I. Valuşescu in [5] provides also the functional models, presented by I. Valuşescu in [6] and by J.A. Ball, N. Cohen in [1]. In the case under consideration it coincides, in essence, with a projection-valued operator function in the representation theory of A.V. Strauss [3], which is closely related to that of M.G Kreĭn. This allows to obtain one more model of a contraction in a reproducing kernel Hilbert space, where values of a kernel are operators acting in the initial Hilbert space.

1 On characteristic function of a contraction

Let $\mathfrak{H}$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, $[\mathfrak{H}]$ be a linear space of linear bounded operators acting in $\mathfrak{H}$ and $[\mathfrak{H}_1, \mathfrak{H}_2]$ be a space of such operators acting from $\mathfrak{H}_1$ to $\mathfrak{H}_2$.

Consider completely nonunitary (c.n.u.) contraction T, that is $\|T\| \leqslant 1$, and there is no subspace of $\mathfrak{H}$, on which T induces a unitary operator. Then, the number 1 is not an eigenvalue of T ($1 \bar{\in} \sigma_p(T)$), hence the operator $(I-T)^{-1}$ exists.

Denote

$$D_T = (I - T^*T)^{\frac{1}{2}}, \;\; D_{T^*} = (I - TT^*)^{\frac{1}{2}}; \quad \mathfrak{D}_T = \overline{D_T\mathfrak{H}}, \;\; \mathfrak{D}_{T^*} = \overline{D_{T^*}\mathfrak{H}}$$

the detect operators and the defect subspaces of T. Then

$$TD_T = D_{T^*}T, \quad T^*D_{T^*} = D_TT^*. \tag{1}$$

Let us form the doubled Hilbert space $\mathcal{H} = \mathfrak{H} \oplus \mathfrak{H}$, denote $\mathcal{H}_1 = \mathfrak{H} \oplus \{0\}$, $\mathcal{H}_2 = \{0\} \oplus \mathfrak{H}$ and $\mathcal{P}_{1,2}$ – orthogonal projections in $\mathcal{H}$ onto its subspaces $\mathcal{H}_{1,2}$, which can be identified with the first and second copies of $\mathfrak{H}$.

Consider operators $\mathcal{V}_0$, $\mathcal{V}$ in $\mathcal{H}$, given on domains $\mathcal{D}(\mathcal{V}_0) = \mathcal{H}_1$, $\mathcal{D}(\mathcal{V}) = \mathcal{H}$ by the block operator matrix

$$\begin{bmatrix} T & 0 \\ D_T & 0 \end{bmatrix}.$$

It is clear that $\mathcal{V}_0$ is an isometry, $\mathcal{V}$ is its partial isometric extension and the dilation of T

$$T^n = \mathcal{P}_1\mathcal{V}^n\mathcal{P}_1, \quad n \geqslant 1,$$

by identifiying $\mathcal{H}_1$ with $\mathfrak{H}$.

It is not hard to see that $\mathcal{V}_0$ and $\mathcal{V}$ are c.n.u. contractions, hence there exist the operators $(\mathcal{I} - \mathcal{V}_0)^{-1}$ and $(\mathcal{I} - \mathcal{V})^{-1}$ ($\mathcal{I}$ is the identity operator in $\mathcal{H}$).

Introduce also the unitary operators in $\mathcal{H}$

$$\mathcal{U} = \begin{bmatrix} D_{T^*} & T \\ -T^* & D_T \end{bmatrix}, \qquad \mathcal{J} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{2}$$

such that $\mathcal{V} = \mathcal{U}\mathcal{J}\mathcal{P}_1$, $\mathcal{J}^2 = \mathcal{I}$, $\mathcal{J}\mathcal{P}_{1,2}\mathcal{J} = \mathcal{P}_{2,1}$. Then the defect operators of a partial isometry $\mathcal{V}$ are the orthogonal projections $D = \mathcal{P}_2$, $D_* = \mathcal{U}\mathcal{P}_1\mathcal{U}^*$ onto the defect subspaces

$$\mathfrak{D} = \left\{ \begin{bmatrix} 0 \\ h \end{bmatrix}, \, h \in \mathfrak{H} \right\}, \quad \mathfrak{D}_* = \mathcal{U}\mathcal{P}_1\mathcal{H} = \left\{ \begin{bmatrix} D_{T^*}h \\ -T^*h \end{bmatrix}, \, h \in \mathfrak{H} \right\}. \tag{3}$$

It is evident that

$$\mathcal{V}D = 0, \qquad \mathcal{V}^*D_* = 0. \tag{4}$$

A maximal function of a contraction T as the operator function $D_{T^*}(I - \omega T^*)^{-1}$, analytic in the unit disk $|\omega| < 1$, was introduced in [5]. Denote $\mathcal{Q}(\omega) = D_*(\mathcal{I} - \omega\mathcal{V}^*)^{-1}$ its analog for $\mathcal{V}$.

Proposition 1 *The Nagy-Foias characteristic function $\Theta_T(\omega)$ of c.n.u. contraction T can be represented as*

$$\Theta_T(\omega) = (\mathcal{U}^* \mathcal{Q}(\omega)|\mathfrak{D})|\, \mathfrak{D}_T. \tag{5}$$

Proof. The operators D_* and $(\mathcal{I} - \omega\mathcal{V}^*)^{-1}$ have the following block matrix representations

$$D_* = \begin{bmatrix} D_{T^*}^2 & -D_{T^*}T \\ -T^*D_{T^*} & T^*T \end{bmatrix},$$

$$(\mathcal{I} - \omega\mathcal{V}^*)^{-1} = \begin{bmatrix} (I - \omega T^*)^{-1} & \omega(I - \omega T^*)^{-1} D_T \\ 0 & I \end{bmatrix},$$

hence

$$\mathcal{Q}(\omega) = \begin{bmatrix} D_{T^*}^2(I - \omega T^*)^{-1} & D_{T^*}\Theta(\omega) \\ -T^*D_{T^*}(I - \omega T^*)^{-1} & -T^*\Theta(\omega) \end{bmatrix}, \tag{6}$$

where $\Theta(\omega) = -T + \omega D_{T^*}(I - \omega T^*)^{-1} D_T$. Then the restriction $\mathcal{U}^*\mathcal{Q}(\omega)|\mathfrak{D}$ is

$$\mathcal{U}^*\mathcal{Q}(\omega)|\mathfrak{D} = \left\{ \begin{bmatrix} \Theta(\omega)h \\ 0 \end{bmatrix},\ h \in \mathfrak{H} \right\}.$$

The identification of $\mathcal{H}_1$ with $\mathfrak{H}$ and the definition of the Nagy-Foias characteristic function as

$$\Theta_T(\omega) = \Theta(\omega)|\mathfrak{D}_T$$

brings to (5). □

Let us first note that the function $\Theta_{\mathcal{V}}(\omega) = \omega\mathcal{Q}(\omega)|\mathfrak{D}$ is the characteristic function of the partial isometry $\mathcal{V}$ in view of (4).

Note also that the formula (5) makes more precise the corresponding formula in [2].

2 On a functional model of T

The properties of the operator function $\mathcal{Q}(\omega)$ are revealed in the following statement.

Proposition 2 *The values of $\mathcal{Q}(\omega)$ are projections in $\mathcal{H}$ onto $\mathfrak{D}_*$. There hold the direct sum decomposition*

$$\mathcal{H} = \operatorname{Ran}(\mathcal{V}_0 - \omega\mathcal{I}) \dotplus \mathfrak{D}_*, \tag{7}$$

and the formula

$$\mathcal{Q}(\omega)\mathcal{V}_0 f_0 = \omega\mathcal{Q}(\omega) f_0, \qquad f_0 \in \mathcal{D}(\mathcal{V}_0). \tag{8}$$

Proof. It follows from (4) that $(\mathcal{I}-\omega\mathcal{V}^*)D_* = D_*$, and also $((\mathcal{I}-\omega\mathcal{V}^*)^{-1}D_* = D_*$, since $\mathcal{I}-\omega\mathcal{V}^*$ is bounded invertible.

Operator D_* is a projection, hence

$$\mathcal{Q}^2(\omega) = D_*(\mathcal{I}-\omega\mathcal{V}^*)^{-1}D_*(\mathcal{I}-\omega\mathcal{V}^*)^{-1} = D_*(\mathcal{I}-\omega\mathcal{V}^*)^{-1} = \mathcal{Q}(\omega).$$

Consider the direct sum decomposition

$$\mathcal{H} = (\mathcal{I}-\omega\mathcal{V}^*)\mathcal{H} = (\mathcal{I}-\omega\mathcal{V}^*)\mathcal{V}\mathcal{H} \dotplus (\mathcal{I}-\omega\mathcal{V}^*)(\mathcal{I}-\mathcal{V})\mathcal{H}. \tag{9}$$

In view of (4) one has

$$\mathcal{Q}(\omega)(\mathcal{I}-\mathcal{V}^*)(\mathcal{I}-\mathcal{V})\mathcal{H} = D_*(\mathcal{I}-\mathcal{V})\mathcal{H} = \mathfrak{D}_*.$$

It is clear from (8) that $\mathcal{Q}(\omega)\mathcal{H} = \mathfrak{D}_*$, so (9) takes the form

$$\mathcal{H} = (\mathcal{I}-\omega\mathcal{V}^*)\mathcal{V}\mathcal{H} \dotplus \mathfrak{D}_*.$$

Now the definitions of $\mathcal{V}$ and $\mathcal{V}_0$ lead to

$$(\mathcal{I}-\omega\mathcal{V}^*)\mathcal{V}\mathcal{H} = (\mathcal{V}-\omega\mathcal{P}_1)\mathcal{H} = \mathrm{Ran}(\mathcal{V}_0-\omega\mathcal{I}),$$

and the relation $\mathcal{Q}(\omega)\mathrm{Ran}(\mathcal{V}_0-\omega\mathcal{I}) = 0$ completes the proof. $\square$

Consider the operator function $\mathcal{Q}_1(\omega) = \mathcal{U}^*\mathcal{Q}(\omega) = \mathcal{P}_1\mathcal{U}^*(\mathcal{I}-\omega\mathcal{V}^*)^{-1}$, which essentially maps $\mathcal{H}$ to $\mathfrak{H}$. Clearly, from (8) one has also

$$\mathcal{Q}_1(\omega)\mathcal{V}_0 f_0 = \omega\mathcal{Q}_1(\omega)f_0, \qquad f_0 \in \mathcal{D}(\mathcal{V}_0). \tag{10}$$

Now we follow to [1] and state some facts presented there. The $\mathcal{H}$-valued function $h(\omega) = \mathcal{Q}_1(\omega)\mathfrak{h}$, $\mathfrak{h} \in \mathcal{H}$ belongs to the Hardy space $H^2_{\mathfrak{H}}$ over the unit disk, and

$$\|h(\omega)\|_{H^2_{\mathfrak{H}}} \leqslant \|\mathfrak{h}\|_{\mathcal{H}},$$

so the map $F_0 : \mathcal{H} \to H^2_{\mathfrak{H}}$ $(F_0\mathfrak{h} = h(\omega))$ is contractive.

Completely nonunitary property of $\mathcal{V}^*$ leads to $\mathrm{Ker}F_0 = \{0\}$. The linear manifold $H_0 = \mathrm{Ran}F_0$ of $H^2_{\mathfrak{H}}$ endowed with the new inner product

$$\langle h(\omega), g(\omega)\rangle_H = \langle \mathfrak{h}, \mathfrak{g}\rangle_{\mathcal{H}} \tag{11}$$

yields the Hilbert space H, and the map F_0 defines a unitary operator $F : \mathcal{H} \to H$, $F^{-1} = F^*$.

Proposition 3 *The operator function $K(\omega,\sigma) = \mathcal{Q}_1(\omega)\mathcal{Q}_1^*(\sigma)$ is a reproducing kerned for the Hilbert space H. In a matrix representation of $K(\omega,\sigma)$ the only nonzero block is*

$$K_{11}(\omega,\sigma) = \left[D_{T_*}(I-\omega T^*)^{-1}(I-\bar{\sigma}T)^{-1}D_{T^*} + \Theta(\omega)\Theta^*(\sigma)\right] \in [\mathfrak{H}]. \tag{12}$$

Proof. The proof is immediate. It is clear that for arbitrary $\mathfrak{g} \in \mathcal{H}$ and fixed σ, $|\sigma| < 1$ one has

$$K(\omega,\sigma)\mathfrak{g} = \mathcal{Q}_1(\omega)\mathcal{Q}_1^*(\sigma)\mathfrak{g} \in H.$$

The use of (11) brings the reproducing property of $K(\omega,\sigma)$

$$\langle h(\sigma),\mathfrak{g}\rangle_{\mathcal{H}} = \langle \mathcal{Q}_1(\sigma)\mathfrak{h},\mathfrak{g}\rangle_{\mathcal{H}} = \langle \mathfrak{h},\mathcal{Q}_1^*(\sigma)\mathfrak{g}\rangle_{\mathcal{H}} = \langle h(\omega),\mathcal{Q}_1(\omega)\mathcal{Q}_1^*(\sigma)\mathfrak{g}\rangle_H = \langle h(\omega),K(\omega,\sigma)\mathfrak{g}\rangle_H,$$

and the formula (12) follows from (6). □

Set $K(\omega,\sigma) = K_{11}(\omega,\sigma)$.

Proposition 4 *In the decomposition*

$$\mathfrak{H} = \mathfrak{D}_{T^*} \oplus \mathfrak{D}_{T^*}^{\perp} \qquad (h = d_* + d_{*\perp}) \tag{13}$$

the kernel $K(\omega,\sigma)$ takes the following form

$$K(\omega,\sigma) = \begin{bmatrix} \dfrac{1}{1-\omega\bar\sigma}\left[I_* - \omega\bar\sigma\Theta_T(\omega)\Theta_T^*(\sigma)\right] & \bigcirc \\ \bigcirc & I_{*\perp} \end{bmatrix} \tag{14}$$

where I_*, $I_{*\perp}$ are identity operators in $\mathfrak{D}_{T^*}$, $\mathfrak{D}_{T^*}^{\perp}$.

Proof. With the use of relations (1) and

$$\omega(I-\omega T^*)^{-1}T^* = (I-\omega T^*)^{-1} - I, \qquad \bar\sigma T(I-\bar\sigma T)^{-1} = (I-\bar\sigma T)^{-1} - I$$

not complicated derivations lead to

$$\begin{aligned}\Theta(\omega)\Theta^*(\sigma) &= \left[-T + \omega D_{T^*}(I-\omega T^*)^{-1}D_T\right]\left[-T^* + \bar\sigma D_T(1-\bar\sigma T)^{-1}D_{T^*}\right] = \\ &= I - (1-\omega\bar\sigma)D_{T^*}(I-\omega T^*)^{-1}(I-\bar\sigma T)^{-1}D_{T^*}.\end{aligned}$$

Thus, the formula (12) can be rewritten as

$$K(\omega,\sigma) = \frac{1}{1-\omega\bar\sigma}\left[I - \Theta(\omega)\Theta^*(\sigma)\right] + \Theta(\omega)\Theta^*(\sigma) = \frac{1}{1-\omega\bar\sigma}\left[I - \omega\bar\sigma\Theta(\omega)\Theta^*(\sigma)\right].$$

Since

$$\Theta_T(\omega) = \Theta(\omega)|\mathfrak{D}_T \in [\mathfrak{D}_T,\mathfrak{D}_{T^*}], \qquad \Theta_T^*(\sigma) = \Theta^*(\sigma)|\mathfrak{D}_{T^*} \in [\mathfrak{D}_{T^*},\mathfrak{D}_T]$$

hence

$$\Theta(\omega)\Theta^*(\sigma)d_* = \Theta_T(\omega)\Theta_T^*(\sigma)d_*.$$

For arbitrary $h \in \mathfrak{H}$ it holds

$$(D_{T^*}d_{*\perp},h) = (d_{*\perp},D_{T^*}h) = 0,$$

so $D_{T^*}d_{*\perp} = 0$, hence

$$\left[I - \Theta(\omega)\Theta^*(\sigma)\right]d_{*\perp} = \left[I - TT^*\right]d_{*\perp} = D_{T_*}^2 d_{*\perp} = 0,$$

and (14) is proved. □

Denote by Ω the multiplication operator by the independent variable ω in H with the domain $\mathcal{D}(\Omega) = \{h(\omega) \in H;\ \omega h(\omega) \in H\}$. It is proved in [4] that $\mathcal{D}(\Omega) = F\mathcal{D}(\mathcal{V}_0)$, so formula (10) can be represented as

$$\mathcal{V}_0 f_0 = F^{-1}\Omega F f_0, \qquad f_0 \in \mathcal{D}(\mathcal{V}_0).$$

Since

$$\mathcal{V}_0 f_0 = \begin{bmatrix} Th \\ D_T h \end{bmatrix}, \quad f_0 = \begin{bmatrix} h \\ 0 \end{bmatrix}, \qquad h \in \mathfrak{H},$$

we get the functional model of c.n.u. contraction T in the form

$$Th = \mathcal{P}_1 F^{-1}\Omega F \begin{bmatrix} h \\ 0 \end{bmatrix}.$$

3 Maximal function $\mathcal{Q}(\omega)$ and the projection-valued function of Strauss

Let A_0 be a closed Hermitian operator in $\mathfrak{H}$ with the domain $\mathcal{D}(A_0)$ not dense in $\mathfrak{H}$, $\overline{\mathfrak{D}(A_0)} \neq \mathfrak{H}$. Assume that A_0 is simple, that is A_0 does not induce a self-adjoint operator on any linear submanifold in $\mathfrak{H}$.

Let some $\gamma \in C^+$ ($\mathrm{Im}\gamma > 0$) be fixed. Then $\mathrm{Ran}(A_0 - \gamma I)$, $\mathrm{Ran}(A_0 - \bar\gamma I)$ are subspaces of $\mathfrak{H}$ and their orthogonal complements

$$\mathfrak{N}_\gamma = \mathfrak{H} \ominus \mathrm{Ran}(A_0 - \bar\gamma I), \qquad \mathfrak{N}_{\bar\gamma} = \mathfrak{H} \ominus \mathrm{Ran}(A_0 - \gamma I)$$

are called the defect subspaces of A_0.

It is known that $\mathcal{D}(A_0) \cap \mathfrak{N}_\gamma = \{0\}$, $\overline{\mathcal{D}(A_0) \dotplus \mathfrak{N}_\gamma} = \mathfrak{H}$, hence the operator A_γ defined on $\mathcal{D}(A_\gamma) = \mathcal{D}(A_0) \dotplus \mathfrak{N}_\gamma$ as

$$A_\gamma f = A_0 f_0 + \gamma f_\gamma, \qquad f_0 \in \mathcal{D}(A_0),\ f_\gamma \in \mathfrak{N}_\gamma$$

is a maximal dissipative extension of A_0, since $\mathrm{Ran}(A_\gamma - \bar\gamma I) = \mathfrak{H}$ (see [3]).

Consider the Cayley transforms of A_0, A_γ

$$\begin{aligned} V_0 = (A_0 - \gamma I)(A_0 - \bar\gamma I)^{-1}, \quad & \mathcal{D}(V_0) = \mathrm{Ran}(A_0 - \bar\gamma I), \\ & \mathrm{Ran}V_0 = \mathrm{Ran}(A_0 - \gamma I), \\ V = (A_\gamma - \gamma I)(A_\gamma - \bar\gamma I)^{-1}, \quad & \mathcal{D}(V) = \mathfrak{H}, \\ & \mathrm{Ran}V = \mathrm{Ran}(A_\gamma - \gamma I) = \mathrm{Ran}(A_0 - \gamma I). \end{aligned}$$

Clearly, operator V is an extension of isometry V_0, and V_0, V are c.n.u. in view of simplicity of A_0, A.

Proposition 5 *The operator V is a partial isometry and* $\mathrm{Ker}V = \mathfrak{N}_\gamma$.

Proof. It is sufficient to show that $\mathrm{Ker}V = \mathfrak{N}_\gamma$. Let $g_\gamma \in \mathfrak{N}_\gamma$ and $(A_\gamma - \bar{\gamma}I)^{-1}g_\gamma = f_0 + f_\gamma$. Then

$$g_\gamma = (A_0 - \bar{\gamma}I)f_0 + (\gamma - \bar{\gamma})f_\gamma$$

implies $f_0 = 0$, so

$$Vg_\gamma = (\gamma - \bar{\gamma})(A_\gamma - \gamma I)f_\gamma = 0.$$

If $Vh = (A_\gamma - \gamma I)(A_\gamma - \bar{\gamma}I)^{-1}h = 0$, then $(A_\gamma - \bar{\gamma}I)^{-1}h = f \in \mathfrak{N}_\gamma$, hence $h = (\gamma - \bar{\gamma})f \in \mathfrak{N}_\gamma$, with the result. □

The converse statement is also true.

Proposition 6 *Let the operator V is a c.n.u. partial isometry. Then its Cayley transform*

$$A = (\gamma I - \bar{\gamma}V)(I - V)^{-1} \tag{15}$$

is a maximal dissipative extension of a simple Hermitian operator

$$A_0 = (\gamma I - \bar{\gamma}V_0)(I - V_0)^{-1}, \qquad V_0 = V|\mathrm{Ker}^{\perp}V. \tag{16}$$

Proof. Since V is c.n.u., the operator $(I-V)^{-1}$ exists, $\mathrm{Ran}(I-V) = \mathcal{D}(A)$ is dense in $\mathfrak{H}$ in view of $\mathrm{Ker}(I - V^*) = \{0\}$. Thus

$$\mathcal{D}(A) = (I-V)\mathfrak{H} = (I-V)\mathrm{Ker}V \dotplus (I-V)\mathrm{Ker}^{\perp}V = \mathrm{Ker}V \dotplus (I-V)\mathrm{Ker}^{\perp}V. \tag{17}$$

If $h_0 \in \mathrm{Ker}V$, so $(I - V)h_0 = h_0$, then also $(1 - V)^{-1}h_0 = h_0$, and

$$Ah_0 = (\gamma I - \bar{\gamma}V)h_0 = \gamma h_0.$$

The operator $V_0 = V|\mathrm{Ker}^{\perp}V$ is a c.n.u. isometry, hence its Cayely transform (16), defined on

$$\mathcal{D}(A_0) = \mathrm{Ran}(I - V_0) = (I - V)\mathrm{Ker}^{\perp}V$$

is Hermitian. Now the formula (17) takes the form

$$\mathcal{D}(A) = \mathcal{D}(A_0) \dotplus \mathrm{Ker}V,$$

hence $Af = A_0f_0 + \gamma h_0$, so A is a dissipative extension of A_0. It follows from

$$(A - \bar{\gamma}I) = (\gamma I - \bar{\gamma}V)(I - V)^{-1} - \bar{\gamma}I = (\gamma - \bar{\gamma})(I - V)^{-1}$$

that $(A - \bar{\gamma}I)\mathcal{D}(A) = (A - \bar{\gamma}I)\mathrm{Ran}(I - V) = \mathfrak{H}$, completing the proof. □

In the base of a representation theory of Strauss it lies the direct sum decomposition

$$\mathfrak{H} = \mathrm{Ran}(A_0 - \lambda I) \dotplus \mathfrak{N}_{\bar{\gamma}}, \qquad \lambda \in C^+, \tag{18}$$

proved in [3]. Formula (18) defines an operator-function $P(\lambda)$ analytic in C^+, which values are skew projections in $\mathfrak{H}$ onto $\mathfrak{N}_{\bar{\gamma}}$ parallel to $\mathrm{Ran}(A_0 - \lambda I)$. Assigning $\mathfrak{N}_{\bar{\gamma}}$-valued function $h(\lambda) = P(\lambda)h$ to each $h \in \mathfrak{H}$, one has the following representation of the Hermitian operator A_0

$$P(\lambda)[A_0 f_0] = \lambda P(\lambda) f_0, \qquad f_0 \in \mathcal{D}(A_0). \tag{19}$$

Now, going back to the decomposition (7) and formula (8), consider the Hermitian Cayley transform $\mathcal{A}_0$ of the isometry $\mathcal{V}_0$ in $\mathcal{H}$, and the linear fractional function $\lambda = \dfrac{\gamma - \bar{\gamma}\omega}{1-\omega}$, mapping the unit disk $|\omega| < 1$ onto the upper half-plane C^+.

The defect subspaces of $\mathcal{A}_0$ denote $\mathcal{N}_\gamma$, $\mathcal{N}_{\bar{\gamma}}$.

Proposition 7 *Decomposition* (7) *coincides with*

$$\mathcal{H} = \mathrm{Ran}(\mathcal{A}_0 - \lambda \mathcal{I}) \dotplus \mathcal{N}_{\bar{\gamma}}. \tag{20}$$

Proof. For the operator $\mathcal{A}_0$ one has

$$\mathcal{D}(\mathcal{A}_0) = \mathrm{Ran}(\mathcal{I} - \mathcal{V}_0) = \left\{ \begin{bmatrix} (I-T)h \\ -D_T h \end{bmatrix},\ h \in \mathfrak{H} \right\},$$

$$\mathrm{Ran}(\mathcal{A}_0) = \left\{ \begin{bmatrix} (\gamma I - \bar{\gamma} T)h \\ -\bar{\gamma} D_T h \end{bmatrix},\ h \in \mathfrak{H} \right\}.$$

Clearly, both

$$\mathrm{Ran}(\mathcal{A}_0 - \bar{\gamma}\mathcal{I}) = \mathcal{D}(\mathcal{V}_0) = \mathcal{P}_1 \mathcal{H}, \qquad \mathrm{Ran}(\mathcal{A}_0 - \gamma \mathcal{I}) = \mathrm{Ran}\mathcal{V}_0 = \mathcal{U}\mathcal{J}\mathcal{P}_1\mathcal{H}$$

are subspaces and their orthogonal complements are

$$\mathcal{N}_\gamma = [\mathcal{D}(\mathcal{V}_0)]^\perp = \mathcal{P}_2 \mathcal{H} = \mathfrak{D}, \qquad \mathcal{N}_{\bar{\gamma}} = \mathrm{Ran}^\perp \mathcal{V}_0 = \mathfrak{D}_*.$$

On account of

$$\mathcal{V}_0 = (\mathcal{A}_0 - \gamma \mathcal{I})(\mathcal{A}_0 - \bar{\gamma} I)^{-1}, \qquad \omega = \frac{\lambda - \gamma}{\lambda - \bar{\gamma}},$$

it is readily seen that

$$\mathcal{V}_0 - \omega \mathcal{I} = \frac{\gamma - \bar{\gamma}}{\lambda - \bar{\gamma}} (\mathcal{A}_0 - \lambda \mathcal{I})(\mathcal{A}_0 - \bar{\gamma}\mathcal{I})^{-1},$$

hence $\mathrm{Ran}(\mathcal{V}_0 - \omega\mathcal{I}) = \mathrm{Ran}(\mathcal{A}_0 - \lambda\mathcal{I})$, since $\mathrm{Ran}(\mathcal{A}_0 - \bar{\gamma}\mathcal{I})^{-1} = \mathcal{D}(\mathcal{A}_0)$. The proof is complete. □

Thus, the maximal function $\mathcal{Q}(\omega)$ and the function of Strauss $\mathcal{P}(\lambda)$ which corresponds to the decomposition (20), are connected by the relation

$$\mathcal{P}(\lambda) = \mathcal{Q}\left(\frac{\lambda - \gamma}{\lambda - \bar{\gamma}} \right).$$

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