Tauberian Theorems by Weighted Summability Method

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Abstract. In this paper, we will show a new Tauberian theorems defined by weighted N"orlund-Ces´aro summability method.

Key Words: Weighted N"orlund-Ces´aro summability; One-sided and two-sided Tauberian conditions.

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Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums $(s_n)$. The $(C,1)$ (see [3],page 7) transform is defined as the n-th partial sum of $(C,1)$ summability and is given by

$$\frac{s_0 + s_1 + \cdots + s_n}{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s \quad \text{as} \quad n \to \infty,$$

then the infinite series $\sum_{n=0}^{\infty} a_n$ is summable to the definite number $s$ by $(C,1)$ method.

Let $\{p_n\}$ be a non-negative, non increasing sequence such that

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty, \quad \text{as} \quad n \to \infty, \quad P_{-1} = p_{-1} = 0.$$

Then the series $\sum_{n=0}^{\infty} a_n$ is said to be almost N"orlund summable to $S$ (or $(N,p_n)$-summable) if

$$\frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \to S,$$

as $n \to \infty$.

The product of $(N,p_n)$ summability and $(C,1)$ summability defines $(N,p_n)(C,1)$ summability and we denote it by $N^p_{n}C_n^1$. Thus if

$$N^p_{n}C_n^1 = \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} s_v \to s \quad \text{as} \quad n \to \infty,$$
where $N^p_n$ denotes the $(N,p_n)$ transform of $s_n$ and $C^1_n$ denotes the $(C,1)$ transform of $s_n$, then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by $(N,p_n)(C,1)$ means or summable $(N,p_n)(C,1)$ to a definite number $s$. The $(N,p_n)$ is a regular method of summability.

$$s_n \to s \Rightarrow C^1_n(s_n) = \frac{1}{n+1} \sum_{k=0}^{n} s_k \to s, \quad \text{as } n \to \infty,$$

$C^1_n$ method is regular,

$$N^p_n(C^1_n(s_n)) = N^p_nC^1_n \to s, \quad \text{as } n \to \infty,$$

$N^p_nC^1_n$ method is regular.

We say that the sequence $(x_n)$ is Nörlund-Cesàro summable to $L$ by the weighted mean method determined by the sequences $(p_n)$, or briefly $(N,p_n)(C,1)$—summable if

$$\lim_{n} N^p_nC^1_n(x) = L. \quad (3)$$

In this case we will write $L = N^p_nC^1_n - \lim_{n} x_n$. We denote by $N^p_nC^1_n$ the set of all sequences which are summable $N^p_nC^1_n$. If

$$\lim_{n} x_n = a \quad (4)$$

eexists, then $(3)$ also exists. However, the converse is not always true. We can show by the following example

**Example 1** Let us consider that $p_n = 1$ for all $n \in \mathbb{N}$. Also we define the following sequence $x = (x_k) = (-1)^k$, then we have

$$\frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{v=0}^{k} (-1)^v \to 0 \quad \text{as } n \to \infty.$$

And as we know $x = (x_k)$, is not convergent.

Notice that $(3)$ may imply $(4)$ under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of sequences follows from its $(N,p_n)(C,1)$—summability and some Tauberian condition is said to be a Tauberian theorem for the $(N,p_n)(C,1)$—summability method.

The theory of Tauberian is extensively studied by many authors([1], [2], [4], [5], [7]). In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions convergence of sequences $(x_n)$, follows from $N^p_nC^1_n$—convergence. This method generalized method given in [5] and [7], it is shown on the following example.

**Example 2** Let us consider that $x_n = n$, then $N^p_nC^1_n$ reduces to the Nörlund method defined in [3] and [7].
1 Main results

In this paper we will generalize Hardy’s Tauberian theorem (see [3]) and obtain new Tauberian theorems for the weighted \((N, p_n)(C, 1)\)–summability method. Let \(u = (u_n)\) be a sequence of real numbers. The classical control modulo of the oscillatory behavior of \((u_n)\) is denoted by \(\omega^{(0)}_n(u) = n\Delta u_n = n(u_n - u_{n-1})\). The general control modulo of the oscillatory behavior of order 1 of \((u_n)\) is defined by

\[
\omega^{(1)}_n(u) = \omega^{(0)}_n(u) - \sigma^{(1)}_n(\omega^{(0)}_n(u)),
\]

where \(\sigma^{(1)}_n(u) = \frac{1}{n+1} \sum_{k=0}^n u_k\). And identity

\[
u_n - \sigma^{(1)}_n(u) = V^{(0)}_n \Delta u,
\]

where \(V^{(0)}_n \Delta u = \frac{1}{n+1} \sum_{k=0}^n k \Delta u_k\), is known as Kronecker identity.

In our case the above definitions are as follows:

\[
\omega^{(1)}_{n,p}(u) = \omega^{(0)}_{n,p}(u) - \sigma^{(1)}_{n,p}(\omega^{(0)}_{n,p}(u)),
\]

where \(\sigma^{(1)}_{n,p}(u) = \frac{1}{P_n} \sum_{k=0}^n p_k \frac{1}{k+1} \sum_{v=0}^k u_v\).

We will start from theorem of Hardy,

**Theorem 1** ([3]) If \((x_n)\) is \((N, p)\) summable to \(x\) and

\[
\omega^{(0)}_{n,p}(x) = 0(1)
\]

then \((x_n)\) converges to \(x\).

**Theorem 2** ([7]) If \((\sigma^{(1)}_{n,p}(u))\) is \((N, p)\) summable to \(s\) and the condition

\[
\omega^{(m)}_{n,p}(u) = 0(1)
\]

holds, then \((u_n)\) converges to \(s\).

Now we are ready to formulate our results which are generalization of the result given above.

**Theorem 3** If

\[
\liminf \frac{P_{t_n}}{P_n} > 1, \quad t > 1 \quad (5)
\]

where \(t_n\), denotes the integer parts of the \([t \cdot n]\) for every \(n \in \mathbb{N}\), and let \((x_k)\) be a sequence of real numbers which converges to \(L\), via \((N, p_n)(C, 1)\)–
summability method. Then \((x_k)\) is convergent to the same number \(L\) if and only if the following two conditions holds:

\[
\sup_{t>1} \liminf_{n} \frac{1}{P_{t_n} - P_n} \sum_{j=n+1}^{t_n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \geq 0 \tag{6}
\]

and

\[
\sup_{0<t<1} \liminf_{n} \frac{1}{P_n - P_{t_n}} \sum_{j=t_n+1}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_n - x_v) \geq 0. \tag{7}
\]

In what follows we will show some auxiliary lemmas which are needful in the sequel.

**Lemma 1** Condition given by relation (5) is equivalent to this one:

\[
\liminf_{n} \frac{P_n}{P_{t_n}} > 1, \quad 0 < t < 1. \tag{8}
\]

**Proof.** We omit it, because it is similar to the lemma 1, given in \[6\]. □

**Proposition 1** Let us suppose that relation (5) is satisfied and let \(x = (x_k)\) be a sequence of complex numbers which is Nörlund-Cesàro summable to \(L\). Then

\[
\lim_{n} \frac{1}{P_{t_n} - P_n} \sum_{j=n+1}^{t_n} p_j \frac{1}{j+1} \sum_{v=0}^{j} x_j = L, \quad \text{for } t > 1 \tag{9}
\]

and

\[
\lim_{n} \frac{1}{P_n - P_{t_n}} \sum_{j=t_n+1}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} x_j = L, \quad \text{for } 0 < t < 1. \tag{10}
\]

**Proof.** (I) Let us consider the case where \(t > 1\). Then we obtain

\[
\frac{1}{P_{t_n} - P_n} \sum_{k=n+1}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v =
\]

\[
= \frac{P_{t_n}}{P_{t_n} - P_n} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{P_n}{P_{t_n} - P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v
\]

\[
= \frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v + \frac{P_n}{P_{t_n} - P_n} \times
\]

\[
\times \left[ \frac{1}{P_{t_n}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right]. \tag{11}
\]
By relation (5) we get
\[ \limsup_{n \to \infty} \frac{P_n}{P_{tn} - P_n} = \frac{1}{\liminf_{n \to \infty} \frac{P_{tn}}{P_n} - 1} < \infty. \]

Now relation (9) follows from relation (11) and assumed convergence of \( N_{pC_n^1}. \)

(II) In this case we have that \( 0 < t < 1. \) Then

\[
\frac{1}{P_n - P_{tn}} \sum_{k=t_n+1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v =
\]

\[
\frac{P_n}{P_n - P_{tn}} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{P_{tn}}{P_n - P_{tn}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v
\]

\[ = \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v + \]

\[ + \frac{P_n}{P_n - P_{tn}} \left[ \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v - \frac{1}{P_{tn}} \sum_{k=0}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right]. \] (12)

Now relation (10), follows from relations (12), (8) and assumed convergence of \( N_{pC_n^1}. \) □

**Proof of Theorem 3**

**Proof.** Necessity. Let us suppose that \( \lim_k x_k = L, \) and \( \lim_n N_{pC_n^1}(x) = L. \)

For every \( t > 1 \) following Proposition 1 we have

\[
\lim_n \frac{1}{P_n - P_{tn}} \sum_{k=n+1}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) =
\]

\[
\lim_n \left\{ \left( \frac{1}{P_n - P_{tn}} \sum_{k=n+1}^{t_n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) - x_n \right\} = 0.
\]

In case where \( 0 < t < 1, \) we find that

\[
\lim_n \frac{1}{P_n - P_{tn}} \sum_{k=t_n+1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v) =
\]

\[
\lim_n \left\{ x_n - \left( \frac{1}{P_n - P_{tn}} \sum_{k=t_n+1}^{n} p_k \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) \right\} = 0.
\]
Sufficient: Assume that conditions (6) and (7) are satisfied. In what follows we will prove that $\lim_{n} x_n = L$. Given any $\epsilon > 0$, by relation (6) we can choose $t_1 > 0$ such that

$$
\liminf_{n} \frac{1}{P_{t_n1} - P_n} \sum_{j=n+1}^{t_n1} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \geq -\epsilon, \quad (13)
$$

where $t_{n_1} = [t_1 \cdot n]$. By the assumed summability $N_{p,C_1}^n$ of $(x_n)$, Proposition 1 for $t > 1$ and taking into account relation (13), we obtain

$$
\limsup_{n} x_n \leq L + \epsilon. \quad (14)
$$

On the other hand, if $0 < t < 1$, for every $\epsilon > 0$, we can choose $0 < t_2 < 1$ such that

$$
\liminf_{n} \frac{1}{P_{n} - P_{t_{n_2}}} \sum_{j=t_{n_2}+1}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_n - x_j) \geq -\epsilon, \quad (15)
$$

where $t_{n_2} = [t_2 \cdot n]$. By the assumed summability $N_{p,C_1}^n$ of $(x_n)$, Proposition 1 for $0 < t < 1$ and relation (15), we get

$$
\liminf_{n} x_n \geq L - \epsilon. \quad (16)
$$

Since $\epsilon > 0$ is arbitrary, combining relations (14) and (16) yields the convergence

$$
\lim_{n} x_n = L.
$$

□

In the next result we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

**Theorem 4** Let us suppose that relation (5) is satisfied. And $(x_n)$ be a sequence of complex numbers, which is $N_{p,C_1}^n$-summable to $L$. Then $(x_n)$ is convergent to the same number $L$ if and only if the following two conditions holds:

$$
\inf_{t>1}\limsup_{n} \left| \frac{1}{P_{n} - P_{t_n}} \sum_{j=n+1}^{t_n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \right| = 0 \quad (17)
$$

and

$$
\inf_{0<t<1}\limsup_{n} \left| \frac{1}{P_{n} - P_{t_n}} \sum_{j=t_{n}+1}^{n} p_j \frac{1}{j+1} \sum_{v=0}^{j} (x_n - x_v) \right| = 0. \quad (18)
$$
Proof. Necessity: Let us suppose that relations (3) and (4) are satisfied. Than by Proposition 1, we get relation (17), for $t > 1$ and relation (18), for $0 < t < 1$.

Sufficient: Let us suppose that relation (5), (3) and (17) are satisfied. Then for any given $\epsilon > 0$, there exists a $t_3 > 1$ such that

$$\limsup_n \left| \frac{1}{P_{t_{n3}} - P_n} \sum_{j=n+1}^{t_{n3}} p_k \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \right| \leq \epsilon,$$

where $t_{n3} = [t_3 \cdot n]$. Taking into account fact that $(x_n)$ is $N^pC^1_n$ summable we get the following estimation

$$\limsup_n |L - x_n| \leq \limsup_n \left| L - \frac{1}{P_{t_{n3}} - P_n} \sum_{j=n+1}^{t_{n3}} p_k \frac{1}{j+1} \sum_{v=0}^{j} x_v \right| + \limsup_n \left| \frac{1}{P_{t_{n3}} - P_n} \sum_{j=n+1}^{t_{n3}} p_k \frac{1}{j+1} \sum_{v=0}^{j} (x_v - x_n) \right| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\lim_n x_n = L$. Second case is similar to the first one and we omit it. □

Remark 1 Theorem 2.3 is generalization of the theorem 2.2, because in the theorem 2.2 are given conditions for Tauberian theorem for Nörlund-Cesàro summability method $(N, p_n)(C, 0)$ and in theorem 2.3 are given conditions for Tauberian theorem for Nörlund-Cesàro summability method $(N, p_n)(C, 1)$.

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