Uniqueness theorem for sequences of piecewise polynomial functions.\footnote{Research is supported by SCS RA grant 15T–1A006}

K. A. Keryan

Abstract. In the paper sequences of piecewise polynomial functions are considered, where each of the function is the projection of subsequent ones. A reconstruction theorem is proved for such sequences converging in measure from its limit if the majorant of the sequence satisfies some condition.

Key Words: piecewise polynomials, uniqueness, majorant, AH-integral

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Introduction

It is well known that there are trigonometric series converging a.e. to 0 and having at least one non-zero coefficient (see e.g. [13, Chapter IX, Theorem 6.14]). One can easily construct such series by Haar, Walsh and Franklin systems.

In the papers [1], [3] uniqueness questions were considered for a.e. converging or summable trigonometric series. Clearly, such questions should be considered under some restrictions.

For Haar series, G.G. Gevorkyan in [5], proved in particular the following theorem

Theorem 1 If the Haar series

$$\sum_{n=1}^{\infty} a_n \chi_n(x)$$

converges a.e. to $f(x)$ and

$$\lim_{\lambda \to \infty} \lambda \mu \{x \in [0, 1]; S^*(x) > \lambda\} = 0,$$

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where \( S^*(x) \) is the majorant of partial sums of the series (1), then the coefficients of series (1) are reconstructed by the following formulas

\[
a_n = \lim_{\lambda \to \infty} \int_0^1 [f(x)]_{\lambda} \chi_n(x) dx,
\]

where

\[
[f(x)]_{\lambda} = \begin{cases} f(x), & \text{for } |f(x)| \leq \lambda \\ 0, & \text{for } |f(x)| > \lambda. \end{cases}
\]

Afterwards this theorem was generalized by V. Kostin in [10] for generalized Haar series and by the author in [9] for generalized Haar series under weaker conditions.

Similar results on uniqueness were also obtained for the Franklin system (see [6],[7]).

Note that partial sums of the series (1) are piecewise constant. Here we are interested in generalization Theorem 1 for piecewise polynomial sequences with (2) replaced by a weaker condition as in [9].

1 Definitions and the main result.

In order to formulate the result let us give some necessary definitions.

Let \( r \in \mathbb{N} \). Denote by \( S_n^{(r)} \) the space of piecewise polynomial functions whose restrictions on each \( \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right) \) for \( 0 \leq k \leq 2^n - 1 \), are polynomials of degree not exceeding \( r \), i.e.

\[
S_n^{(r)} = \left\{ f; \deg(f|_{\left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right)}) \leq r \text{ for } 0 \leq k \leq 2^n - 1 \right\}.
\]

Let \( P_n^{(r)} : L[0,1] \to S_n^{(r)} \) be the orthogonal projection, i.e.

\[
(f, g) = (P_n^{(r)} f, g) \text{ for all } f \in L[0,1], g \in S_n^{(r)}.
\]

Let the sequence of functions \( (S_n)_{n \geq 0} \) satisfy \( S_n \in S_n^{(r)} \) for \( n \geq 0 \) and

\[
P_n^{(r)}(S_m) = S_n \text{ for } m \geq n. \tag{3}
\]

Set

\[
S^*(x) = \sup_n |S_n(x)|.
\]

We denote by \( \mathcal{D} \) the set of all dyadic intervals, i.e.

\[
\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n, \text{ where } \mathcal{D}_n = \left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right) ; 0 \leq k \leq 2^n - 1 \right\}.
\]

We call an interval \( I \in \mathcal{D}_n \) an interval of rank \( n \) and set \( r(I) := n \).
Let functions $h_m(x), h_m : [0, 1] \to R$ satisfy the following conditions:

\[ \begin{align*}
\text{(i)} & \quad 0 \leq h_1(x) \leq h_2(x) \leq \cdots \leq h_m(x) \leq \cdots, \quad \lim_{m \to \infty} h_m(x) = \infty, \\
\text{(ii)} & \quad \text{there exist a constant } C > 0 \text{ and intervals } I^m_1, \ldots, I^m_{n_m} \in \mathcal{D}, \text{ so that} \\
& \quad I^m_i \cap I^m_j = \emptyset, \quad i \neq j, \quad \bigcup_{k=1}^{n_m} I^m_k = [0, 1), \quad \text{and} \\
& \quad \sup_{x \in I^m_k} h_m(x) \leq C \inf_{x \in I^m_k} h_m(x), \\
\text{(iii)} & \quad \inf_{m,k} \int_{I^m_k} h_m(x) dx > 0.
\end{align*} \]  

In other words, for any function $h_m$ the interval $[0, 1]$ can be split into small dyadic intervals, so that the supremum and infimum of that function on each interval are comparable and integrals over that intervals are bigger than some positive constant.

**Theorem 2** Let the functions $h_m(x)$ satisfy conditions (4), (5), (6). If the sequence $(S_n)$ satisfying (3) converges in measure to a function $S$ and

\[ \lim_{m \to \infty} \int_{\{x \in [0,1]; S^*(x) > h_m(x)\}} h_m(x) dx = 0 \]  

then for any $g \in S^{(r)}_n$,

\[ (S_n, g) = \lim_{m \to \infty} \int_0^1 [S(x)]_{h_m(x)} g(x) dx. \]

This theorem actually enables to recover the sequence $(S_n)$ from its limit $S$ under mentioned conditions. Generally speaking the limit may be not Lebesgue integrable.

2 Proof of the main theorem.

We need the following simple lemma.

**Lemma 1** Let $P$ be a polynomial of degree not exceeding $r$ on $[\alpha, \beta]$ and

\[ l = \max_{t \in [\alpha, \beta]} |P(t)|, \]

then

\[ \mu \left\{ t \in [\alpha, \beta]; |P(t)| > \frac{l}{2} \right\} \geq \frac{\beta - \alpha}{4r^2}. \]
It follows from (5), (6) that 
\[ \varepsilon \]
Denote 
\[
\text{for any } t \in [\alpha, \beta].
\]
Therefore if 
\[ m \text{ sufficiently large} \]
for any \( t \in [t_0 - \frac{\beta - \alpha}{4r^2}, t_0 + \frac{\beta - \alpha}{4r^2}] \cap [\alpha, \beta] \). This yields the desired estimate. □

**Proof of Theorem 2.** Without loss of generality we can assume that \( g \) is a polynomial of degree no more than \( r \). Indeed, if the theorem is true for any such polynomial, then for any \( g \in S_n^{(r)} \) there exist polynomials \( P_k, k = 1, \ldots, 2^n \) so that 
\[
g = \sum_{k=1}^{2^n} P_k \cdot \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})},
\]
hence applying the theorem for each \( P_k \) we get
\[
(S_n, g) = \sum_{k=1}^{2^n} (S_n, P_k \cdot \mathbf{1}_{[\frac{k-1}{2^n}, \frac{k}{2^n})}) =
\]
\[
= \sum_{k=1}^{2^n} \lim_{m \to \infty} \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} [S(x)]_{h_m(x)} \cdot P_k(x) \, dx = \lim_{m \to \infty} \int_{0}^{1} [S(x)]_{h_m(x)} \cdot g(x) \, dx.
\]
Denote 
\[
\lambda_k^m = \inf_{x \in I_k^m} h_m(x) \quad \text{and} \quad \varepsilon_0 = \inf_{m,k} \lambda_k^m \mu(I_k^m).
\]
It follows from (5), (6) that \( \varepsilon_0 > 0 \). Take \( \varepsilon < \varepsilon_0/(4r^2) \). It follows from (7) that for sufficiently large \( m \) we have
\[
\int_{E_m} h_m(x) \, dx < \varepsilon, \quad \text{where } E_m = \{ x \in [0, 1]; S^*(x) > h_m(x) \}. \tag{8}
\]
Hence we get from (5)
\[
\sum_{k=1}^{n_m} \lambda_k^m \mu \{ x \in I_k^m, S^*(x) > C\lambda_k^m \} < \varepsilon. \tag{9}
\]
Fix \( m \) and denote by \( k_0 \) the maximal rank of the intervals \( I_k^m \), i.e.
\[
k_0 = \max_{1 \leq k \leq n_m} r(I_k^m).
\]
Denote \( \tilde{S}_{k_0}(x) = S_{r(I_k^m)}(x) \), for \( x \in I_k^m \). It is not hard to check that \( |\tilde{S}_{k_0}(x)| \leq 2C\lambda_k^m \), when \( x \in I_k^m \). Indeed, assume, to the contrary, that there exist a \( k', 1 \leq k' \leq n_m \), and a point \( x_0 \in I_{k'}^m \) such that \( |\tilde{S}_{k_0}(x_0)| > 2C\lambda_{k'}^m \). Then applying Lemma 4 we get
\[
\mu \{ t \in I_{k'}^m; |S^*(t)| > C\lambda_{k'}^m \} \geq \mu \{ t \in I_{k'}^m; |\tilde{S}_{k_0}(t)| > C\lambda_k^m \} \geq \frac{\mu(I_k^m)}{4r^2}.
\]
Therefore it follows from (9) and the definition of \( \varepsilon_0 \) that
\[
4r^2 \varepsilon > 4r^2 \lambda_k^m \mu \{ t \in I_k^m; |S^*(t)| > C\lambda_k^m \} > \lambda_k^m \mu(I_k^m) \geq \varepsilon_0,
\]
which contradicts to the choice of \( \varepsilon \).

Let \( I_1^m \) be the union of the intervals \( I_1, I_2 \) of the rank \( r(I_1^m) + 1 \). If \( |S_{r(I_1^m)+1}(x)| \leq 2C\lambda_1^m \) for any \( x \in I_1^m \), then we will set \( \tilde{S}_{k_0+1}(x) = S_{r(I_1^m)+1}(x) \) on \( I_1^m \) and call each of the intervals \( I_1, I_2 \) the 1st class intervals for \( \tilde{S}_{k_0+1}(x) \). Otherwise we will set \( \tilde{S}_{k_0+1}(x) = S_{r(I_1^m)}(x) \) on \( I_1^m \), and call \( I_1^m \) the 2nd class interval for \( \tilde{S}_{k_0+1}(x) \). Similarly we can define the class of intervals \( I_2^m, \ldots, I_{n_{\max}}^m \), and determine \( \tilde{S}_{k_0+1}(x) \) on each of \( I_2^m, \ldots, I_{n_{\max}}^m \) intervals.

Assuming that \( \tilde{S}_{k_0+t}(x) \) is defined, determine \( \tilde{S}_{k_0+t+1}(x) \) as follows. The intervals of 2nd class for \( \tilde{S}_{k_0+t}(x) \) will be intervals of 2nd class for \( \tilde{S}_{k_0+t+1}(x) \) as well and let us set \( \tilde{S}_{k_0+t+1}(x) = \tilde{S}_{k_0+t}(x) \) on these intervals. If \( I \) is an interval of 1st class for \( \tilde{S}_{k_0+t}(x) \), then we act as follows. Let \( I \) be the union of intervals \( I_1, I_2 \) of the rank \( r(I) + 1 \). Without loss of generality we can assume that \( I \subset I_1^m \). If \( S_{r(I)+1}(x) \leq 2C\lambda_1^m \), for \( x \in I \) then we will set \( \tilde{S}_{k_0+t+1}(x) = S_{r(I)+1}(x) \) on \( I \), and each of the intervals \( I_1, I_2 \) will be called interval of 1st class for \( \tilde{S}_{k_0+t+1}(x) \). Otherwise we will call the interval \( I \) the 2nd class interval for \( \tilde{S}_{k_0+t+1}(x) \), and set \( \tilde{S}_{k_0+t+1}(x) = S_{r(I)}(x) \) for \( x \in I \).

So the function \( \tilde{S}_{k_0+t}(x) \) is a polynomial of degree not exceeding \( r \) on intervals \( I_1, \ldots, I_t \) (generally speaking, the ranks of the intervals \( I_s, s = 1, \ldots, t \) may vary depending on \( s \) ) and

\[
\tilde{S}_{k_0+t}(x) = S_{r(I_j)}(x) \quad \text{for} \quad x \in I_j. \tag{10}
\]

It follows from the definition of \( \tilde{S}_{k_0+t}(x) \) that

\[
|\tilde{S}_{k_0+t}(x)| \leq 2C\lambda_1^m, \quad \text{for} \quad x \in I_k^m. \tag{11}
\]

Denote by \( A_{k_0,t} \) the union of all intervals of 2nd class for \( \tilde{S}_{k_0+t}(x) \), and let \( A_{k_0,t}^k = A_{k_0,t} \cap I_k^m \). Let us prove

\[
\mu(A_{k_0,t}^k) \leq 8r^2 \mu \{ x \in I_k^m : S^*(x) > C\lambda_k^m \}. \tag{12}
\]

Note that the set \( A_{k_0,t}^k \) is the union of all intervals of 2nd class for \( \tilde{S}_{k_0+t}(x) \), which are subsets of \( I_k^m \). Therefore each of these intervals \( I \) contains at least one interval \( J_t \), such that \( r(J_t) = r(I) + 1 \) and \( |S_{r(J_t)}(x_0)| > 2C\lambda_k^m \), for some \( x_0 \in J_t \). Therefore applying Lemma [1] we get

\[
\mu \{ t \in J_t : |S^*(t)| > C\lambda_k^m \} \geq \mu \{ t \in J_t : |S_{r(J)}(t)| > C\lambda_k^m \} \geq \frac{\mu(J_t)}{4r^2}.
\]

Clearly \( \mu(I) = 2\mu(J_t) \), hence we obtain

\[
\mu(A_{k_0,t}^k) = \sum \mu(I) \leq 2 \sum \mu(J_t) \leq 8r^2 \mu \{ x \in I_k^m : S^*(x) > C\lambda_k^m \}.
\]
Let \( P \) be a polynomial of degree not exceeding \( r \) and \( M = \max_{t \in [0, 1]} |P(t)| \). Notice that it follows from (3) that \( (S_0, P) = \sum_{k=1}^{n_m} \int_{I_k^m} \tilde{S}_{k_0+l}(x) P(x) \, dx \). Now let us estimate the following expression:

\[
|S_0, P| - ([S]_{h_m}, P) \leq \sum_{k=1}^{n_m} \left| \int_{I_k^m} \tilde{S}_{k_0+l}(x) P(x) \, dx - \int_{I_k^m \setminus E_m} S(x) P(x) \, dx \right| + \int_{E_m} h_m(x) |P(x)| \, dx \leq \sum_{k=1}^{n_m} \left| \int_{(I_k^m \setminus E_m) \cap A_{k_0,t}} \left( \tilde{S}_{k_0+l}(x) - S(x) \right) P(x) \, dx \right| + \int_{E_m} \left| \tilde{S}_{k_0+t}(x) P(x) \right| \, dx + M \sum_{k=1}^{n_m} \int_{(I_k^m \setminus E_m) \cap A_{k_0,t}} \left( \left| \tilde{S}_{k_0+l}(x) \right| + |S(x)| \right) \, dx + M \int_{E_m} h_m(x) \, dx = I_1 + I_2 + I_3 + I_4.
\]

It follows from (11) and the definition of \( \lambda_k^m \) that

\[
I_2 \leq 2CM \int_{E_m} h_m(x) \, dx = 2CI_4,
\]

therefore we get from (8)

\[
I_2 + I_4 < (2C + 1)M \varepsilon. \tag{13}
\]

Since \( S_n \) converges in measure to \( S \), we obtain for a.a. \( x \in I_k^m \setminus E_m \)

\[
|S(x)| \leq S^* (x) \leq h_m(x) \leq C \lambda_k^m. \tag{14}
\]

Hence we get from (11) and (12)

\[
I_3 \leq 4CM \sum_{k=1}^{n_m} \lambda_k^m \mu(A_{k_0,t}) \leq 32CMr^2 \sum_{k=1}^{n_m} \lambda_k^m \mu \{ x \in I_k^m; S^*(x) > C \lambda_k^m \},
\]

and applying (9) we obtain the following estimate for \( I_3 \)

\[
I_3 \leq 32CMr^2 \varepsilon. \tag{15}
\]

It remains to estimate \( I_1 \). Since \( S_n \) converges in measure to \( S \), there exists \( l_0 \), such that for any \( l' > l_0 \) we have

\[
\mu \{ x \in I_k^m; |S_{k_0+l'}(x) - S(x)| \geq \varepsilon \} < \frac{\varepsilon}{\lambda_k^m n_m}. \tag{16}
\]

Let us choose \( l \) so that \( r(I_k^m) + l > k_0 + l_0 \), for any \( k = 1, 2, \ldots n_m \). Denote by

\[
B_{k_0,t}^k = \{ x \in I_k^m; |S_{r(I_k^m)+l}(x) - S(x)| \geq \varepsilon \}, \quad B_{k_0,t} = \bigcup_{k=1}^{n_m} B_{k_0,t}^k.
\]
Note that it follows from (16) that \( \mu(B_{k_0,l}^k) < \varepsilon/(\lambda_k^m n_m) \), hence, taking into account \( \tilde{S}_{k_0+l}(x) = S_{r(I_k^m)}(x) \), for \( x \in I_k^m \cap A_{k_0,l}^k \), and the inequalities (11), (14), we obtain

\[
I_1 \leq M \sum_{k=1}^{n_m} \int \left( |\tilde{S}_{k_0+l}(x) - S(x)| \right) dx +
M \sum_{k=1}^{n_m} \int \left( |\tilde{S}_{k_0+l}(x)| + |S(x)| \right) dx \leq
\leq M \varepsilon + \sum_{k=1}^{n_m} 4CM \lambda_k^m \frac{\varepsilon}{\lambda_k^m n_m}.
\]

So we get \( I_1 \leq (4C + 1)M \varepsilon \). Combining this estimate with the estimates (13), (15), we get for sufficiently large \( m \)

\[
|(S_0, P) - ([S]_{h_m}, P)| < (6C + 2 + 32Cr^2)M \varepsilon.
\]

Theorem 2 is proved.

References


Karen Keryan
Yerevan State University,
American University of Armenia
karenkeryan@ysu.am kkeryan@aua.am

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