\textbf{$n$-Points Inequalities of Hermite-Hadamard Type for $h$-Convex Functions on Linear Spaces}

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Abstract. Some $n$-points inequalities of Hermite-Hadamard type for $h$-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

Key Words: Convex functions, Integral inequalities, $h$-Convex functions.

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1 Introduction

We recall here some concepts of convexity that are well known in the literature.

Let $I$ be an interval in $\mathbb{R}$.

Definition 1 (\cite{26}) We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that $f$ belongs to the class $Q(I)$ if $f$ is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y). \quad (1)$$

Some further properties of this class of functions can be found in \cite{20}, \cite{21}, \cite{23}, \cite{32}, \cite{35} and \cite{39}. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where $C$ is a convex subset of the real or complex linear space $X$ and the inequality (1) is satisfied for any vectors $x, y \in C$ and $t \in (0, 1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then it is of Godunova-Levin type.
Definition 2 ([23]) We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

(2)

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contains all nonnegative monotone, convex and quasi convex functions, i.e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

(3)

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on $P$-functions see [23] and [33] while for quasi convex functions, the reader can consult [22].

If $f : C \subseteq X \to [0, \infty)$, where $C$ is a convex subset of the real or complex linear space $X$, then we say that it is of $P$-type (or quasi-convex) if the inequality (2) (or (3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([7]) Let $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be $s$-convex (in the second sense) or Breckner $s$-convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [18], [19], [27], [29] and [38].

The concept of Breckner $s$-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \| \cdot \|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \geq 1$ is convex on $X$.

Utilising the elementary inequality $(a + b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0, 1]$, we have for the function $g(x) = \|x\|^s$ that

$$g(tx + (1-t)y) = \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s$$

$$\leq t^s g(x) + (1-t)^s g(y)$$

for any $x, y \in X$ and $t \in [0, 1]$ which shows that $g$ is Breckner $s$-convex on $X$.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of $h$-convex functions as follows.

Assume that $f$ and $J$ are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions $h$ and $f$ are real non-negative functions defined in $J$ and $I$, respectively.
Definition 4 ([41]) Let $h : J \to [0, \infty)$ with $h$ not identical to 0. We say that $f : I \to [0, \infty)$ is an $h$-convex function if for all $x, y \in I$ we have

$$f (tx + (1 - t)y) \leq h(t)f(x) + h(1-t)f(y)$$ (4)

for all $t \in (0,1)$.

For some results concerning this class of functions see [41], [6], [30], [39], [37] and [40].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval $I$ be the corresponding convex subset $C$ of the linear space $X$.

We can introduce now another class of functions.

Definition 5 We say that the function $f : C \subseteq X \to [0, \infty)$ is of $s$-Godunova-Levin type, with $s \in [0,1]$, if

$$f (tx + (1 - t)y) \leq tsf(x) + (1-t)sf(y),$$ (5)

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of $P$-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of $s$-Godunova-Levin functions defined on $C$, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

The following inequality holds for any convex function $f$ defined on $\mathbb{R}$

$$(b - a)f \left( \frac{a + b}{2} \right) < \int_a^b f(x)dx < (b - a)\frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}. $$ (6)

It was firstly discovered by Ch. Hermite in 1881 in the journal Mathesis (see [31]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite’s result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite’s note in Mathesis [31]. Since [6] was known as Hadamard’s inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[13], [24] and [34].

We can state the following generalization of the Hermite-Hadamard inequality for $h$-convex functions defined on convex subsets of linear spaces [17].
Theorem 1 Assume that the function $f : C \subseteq X \to [0, \infty)$ is a $h$-convex function with $h \in L[0, 1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f [(1 - t) x + ty]$ is Lebesgue integrable on $[0, 1]$. Then
\[
\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{x + y}{2} \right) \leq \int_{0}^{1} f [(1 - t) x + ty] dt \leq [f(x) + f(y)] \int_{0}^{1} h(t) dt. \tag{7}
\]

Remark 1 If $f : I \to [0, \infty)$ is a $h$-convex function on an interval $I$ of real numbers with $h \in L[0, 1]$ and $f \in L[a, b]$ with $a, b \in I$, $a < b$, then from (7) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in [37]
\[
\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(u) du \leq [f(a) + f(b)] \int_{0}^{1} h(t) dt. \tag{9}
\]

If we write (7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions
\[
f \left( \frac{x + y}{2} \right) \leq \int_{0}^{1} f [(1 - t) x + ty] dt \leq \frac{f(x) + f(y)}{2}. \tag{8}
\]

If we write (7) for the case of $P$-type functions $f : C \to [0, \infty)$, i.e., $h(t) = 1$, $t \in [0, 1]$, then we get the inequality
\[
\frac{1}{2} f \left( \frac{x + y}{2} \right) \leq \int_{0}^{1} f [(1 - t) x + ty] dt \leq f(x) + f(y), \tag{9}
\]
that has been obtained for functions of real variable in [23].

If $f$ is Breckner $s$-convex on $C$, for $s \in (0, 1)$, then by taking $h(t) = t^s$ in (7) we get
\[
2^{s-1} f \left( \frac{x + y}{2} \right) \leq \int_{0}^{1} f [(1 - t) x + ty] dt \leq \frac{f(x) + f(y)}{s + 1}, \tag{10}
\]
that was obtained for functions of a real variable in [18].

Since the function $g(x) = \|x\|^s$ is Breckner $s$-convex on the normed linear space $X$, $s \in (0, 1)$, then for any $x, y \in X$ we have
\[
\frac{1}{2} \|x + y\|^s \leq \int_{0}^{1} \|(1 - t) x + ty\|^s dt \leq \frac{\|x\|^s + \|y\|^s}{s + 1}. \tag{11}
\]

If $f : C \to [0, \infty)$ is of $s$-Godunova-Levin type, with $s \in [0, 1)$, then
\[
\frac{1}{2^{s+1}} f \left( \frac{x + y}{2} \right) \leq \int_{0}^{1} f [(1 - t) x + ty] dt \leq \frac{f(x) + f(y)}{1 - s}. \tag{12}
\]
We notice that for \( s = 1 \) the first inequality in (12) still holds, i.e.

\[
\frac{1}{4} f \left( \frac{x + y}{2} \right) \leq \int_0^1 f \left[ (1 - t) x + ty \right] dt.
\]

The case of functions of real variables was obtained for the first time in [23].

Motivated by the above results, in this paper some \( n \)-points inequalities of Hermite-Hadamard type for \( h \)-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

## 2 Some New Results

In [17] we also obtained the following result:

**Theorem 2** Assume that the function \( f : C \subseteq X \to [0, \infty) \) is an \( h \)-convex function with \( h \in L \mathbb{[0,1]} \). Let \( y, x \in C \) with \( y \neq x \) and assume that the mapping \( [0,1] \ni t \mapsto f \left[ (1 - t) x + ty \right] \) is Lebesgue integrable on \([0,1]\). Then for any \( \lambda \in [0,1] \) we have the inequalities

\[
\frac{1}{2h \left( \frac{1}{2} \right)} \left\{ (1 - \lambda) f \left[ \frac{(1 - \lambda) x + (\lambda + 1) y}{2} \right] + \lambda f \left[ \frac{(2 - \lambda) x + \lambda y}{2} \right] \right\} 
\leq \int_0^1 f \left[ (1 - t) x + ty \right] dt
\leq [f ((1 - \lambda) x + \lambda y) + (1 - \lambda) f (y) + \lambda f (x)] \int_0^1 h (t) dt
\leq \{[h (1 - \lambda) + \lambda] f (x) + [h (\lambda) + 1 - \lambda] f (y)\} \int_0^1 h (t) dt.
\]

We can state the following new corollary as well:

**Corollary 1** With the assumptions of Theorem 2 we have

\[
\frac{1}{2h \left( \frac{1}{2} \right)} \times \int_0^1 (1 - \lambda) \left\{ f \left[ \frac{(1 - \lambda) x + (\lambda + 1) y}{2} \right] + f \left[ \frac{(1 - \lambda) y + (\lambda + 1) x}{2} \right] \right\} d\lambda
\leq \int_0^1 f \left[ (1 - t) x + ty \right] dt
\leq \left[ \int_0^1 f ((1 - \lambda) x + \lambda y) d\lambda + \frac{f (y) + f (x)}{2} \right] \int_0^1 h (t) dt
\leq [f (x) + f (y)] \left[ \int_0^1 h (\lambda) d\lambda + \frac{1}{2} \right] \int_0^1 h (t) dt.
\]
Proof. The proof follows by integrating the inequality (14) over \( \lambda \) and by using the equality
\[
\int_{0}^{1} \lambda f \left( \frac{(2 - \lambda) x + \lambda y}{2} \right) d\lambda = \int_{0}^{1} (1 - \mu) f \left( \frac{(1 + \mu) x + (1 - \mu) y}{2} \right) d\mu.
\]
□

The following result for double integral also holds:

Corollary 2 With the assumptions of Theorem 2 we have
\[
\frac{1}{2h \left( \frac{1}{2} \right) (b - a)^2} \int \int_{\alpha \beta} \left\{ f \left( \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} \right) + f \left( \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right) \right\} d\alpha d\beta
\leq \int_{0}^{1} f \left( (1 - t) x + ty \right) dt
\leq \left[ \frac{1}{(b - a)^2} \int \int_{\alpha \beta} h \left( \frac{\beta}{\alpha + \beta} \right) d\alpha d\beta + \frac{1}{2} \right] [f(x) + f(y)] \int_{0}^{1} h(t) dt,
\]
for any \( b > a \geq 0 \).

Proof. If we take \( \lambda = \frac{\alpha}{\alpha + \beta} \) we have
\[
\frac{1}{2h \left( \frac{1}{2} \right)} \int \int_{\alpha \beta} \left\{ \frac{\beta}{\alpha + \beta} f \left( \frac{\beta x + (2\alpha + \beta) y}{2(\alpha + \beta)} \right) + \frac{\alpha}{\alpha + \beta} f \left( \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right) \right\}
\leq \int_{0}^{1} f \left( (1 - t) x + ty \right) dt
\leq \left[ h \left( \frac{\beta}{\alpha + \beta} \right) + \frac{\beta}{\alpha + \beta} f(y) + \frac{\alpha}{\alpha + \beta} f(x) \right] \int_{0}^{1} h(t) dt
\leq \left\{ h \left( \frac{\beta}{\alpha + \beta} \right) + \frac{\alpha}{\alpha + \beta} \right\} f(x) + \left\{ h \left( \frac{\alpha}{\alpha + \beta} \right) + \frac{\beta}{\alpha + \beta} \right\} f(y)
\times \int_{0}^{1} h(t) dt,
\]
for any \( \alpha, \beta \geq 0 \) with \( \alpha + \beta > 0 \).
Since the mapping \([0, 1] \ni t \mapsto f [(1 - t) x + ty] \) is Lebesgue integrable on \([0, 1] \), then the double integral \( \int_a^b \int_a^b f \left( \frac{\beta x + \gamma y}{\alpha + \beta} \right) \, d\alpha d\beta \) exists for any \( b > a \geq 0 \).

The same holds for the other integrals in (16).

Integrating the inequality (17) on the square \([a, b]^2 \) over \((\alpha, \beta)\) we have

\[
\frac{1}{2h \left( \frac{1}{2} \right) (b - a)^2} \int_a^b \int_a^b \left\{ \frac{\beta}{\alpha + \beta} f \left[ \frac{\beta x + (2\alpha + \beta) y}{2 (\alpha + \beta)} \right] + \frac{\alpha}{\alpha + \beta} f \left[ \frac{(2\beta + \alpha) x + \alpha y}{2 (\alpha + \beta)} \right] \right\} \, d\alpha d\beta \leq \int_0^1 f [(1 - t) x + ty] \, dt
\]

\[
\leq \int_a^b \int_a^b \left\{ \frac{\beta}{\alpha + \beta} f \left( \frac{\beta x + \alpha y}{\alpha + \beta} \right) + \frac{\alpha}{\alpha + \beta} f (y) + \frac{\alpha}{\alpha + \beta} f (x) \right\} \, d\alpha d\beta \int_0^1 h (t) \, dt
\]

\[
\leq \frac{1}{(b - a)^2} \int_0^1 h (t) \, dt \times \int_a^b \int_a^b \left\{ \frac{\alpha}{\alpha + \beta} f \left( \frac{\alpha x + (2\beta + \alpha) y}{2 (\alpha + \beta)} \right) + \frac{\beta}{\alpha + \beta} f (y) \right\} \, d\alpha d\beta. \quad (18)
\]

Observe that

\[
\int_a^b \int_a^b \frac{\beta}{\alpha + \beta} f \left[ \frac{\beta x + (2\alpha + \beta) y}{2 (\alpha + \beta)} \right] \, d\alpha d\beta = \int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} f \left[ \frac{\alpha x + (2\beta + \alpha) y}{2 (\alpha + \beta)} \right] \, d\alpha d\beta
\]

and then

\[
\int_a^b \int_a^b \left\{ \frac{\beta}{\alpha + \beta} f \left[ \frac{\beta x + (2\alpha + \beta) y}{2 (\alpha + \beta)} \right] + \frac{\alpha}{\alpha + \beta} f \left[ \frac{(2\beta + \alpha) x + \alpha y}{2 (\alpha + \beta)} \right] \right\} \, d\alpha d\beta
\]

\[
= \int_a^b \int_a^b \left\{ f \left( \frac{\alpha x + (2\beta + \alpha) y}{2 (\alpha + \beta)} \right) + f \left( \frac{(2\beta + \alpha) x + \alpha y}{2 (\alpha + \beta)} \right) \right\} \, d\alpha d\beta.
\]

Also

\[
\int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} \, d\alpha d\beta = \int_a^b \int_a^b \frac{\beta}{\alpha + \beta} \, d\alpha d\beta
\]

and since

\[
\int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} \, d\alpha d\beta + \int_a^b \int_a^b \frac{\beta}{\alpha + \beta} \, d\alpha d\beta = \int_a^b \int_a^b \frac{\alpha + \beta}{\alpha + \beta} \, d\alpha d\beta = (b - a)^2,
\]

then we have

\[
\int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} \, d\alpha d\beta = \frac{1}{2} (b - a)^2.
\]
Moreover, we have
\[ \int_{a}^{b} \int_{a}^{b} h \left( \frac{\alpha}{\alpha + \beta} \right) \, d\alpha d\beta = \int_{a}^{b} \int_{a}^{b} h \left( \frac{\beta}{\alpha + \beta} \right) \, d\alpha d\beta. \]

Utilising (18), we get the desired result (16).
\[ \square \]

**Remark 2** Let \( f : C \subseteq X \to \mathbb{C} \) be a convex function on the convex subset \( C \) of a real or complex linear space \( X \). Then for any \( x, y \in C \) and \( b > a \geq 0 \) we have
\[
f \left( \frac{x + y}{2} \right) \leq \frac{1}{(b - a)^2} \times \int_{a}^{b} \int_{a}^{b} \frac{\alpha}{\alpha + \beta} \left\{ f \left[ \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} \right] + f \left[ \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right] \right\} \, d\alpha d\beta
\leq \int_{0}^{1} f \left[ (1 - t) x + ty \right] \, dt
\leq \frac{1}{2} \left[ \frac{1}{(b - a)^2} \int_{a}^{b} \int_{a}^{b} f \left( \frac{\beta x + \alpha y}{\alpha + \beta} \right) \, d\alpha d\beta + \frac{f(y) + f(x)}{2} \right]
\leq \frac{f(y) + f(x)}{2}.
\]

The second and third inequalities are obvious from (16) for \( h(t) = t \).

By the convexity of \( f \) we have
\[
\frac{1}{2} \left\{ f \left[ \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} \right] + f \left[ \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right] \right\}
\geq f \left[ \frac{1}{2} \left\{ \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} + \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right\} \right]
= f \left( \frac{x + y}{2} \right)
\]
for any \( \alpha, \beta \geq 0 \) with \( \alpha + \beta > 0 \).

If we multiply this inequality by \( \frac{2a}{a + \beta} \geq 0 \) and integrate on the square \([a, b]^2\) we get
\[
\int_{a}^{b} \int_{a}^{b} \frac{\alpha}{\alpha + \beta} \left\{ f \left[ \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} \right] + f \left[ \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right] \right\} \, d\alpha d\beta
\geq 2f \left( \frac{x + y}{2} \right) \int_{a}^{b} \int_{a}^{b} \frac{\alpha}{\alpha + \beta} \, d\alpha d\beta = (b - a)^2 f \left( \frac{x + y}{2} \right),
\]
since we know that
\[
\int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} d\alpha d\beta = \frac{1}{2} (b - a)^2.
\]
This proves the first inequality in (19).
By the convexity of \( f \) we also have
\[
f \left( \frac{\beta x + \alpha y}{\alpha + \beta} \right) \leq \frac{\beta}{\alpha + \beta} f(x) + \frac{\alpha}{\alpha + \beta} f(y)
\]
for any \( \alpha, \beta \geq 0 \) with \( \alpha + \beta > 0 \). Integrating on the square \([a, b]^2\) we get
\[
\int_a^b \int_a^b f \left( \frac{\beta x + \alpha y}{\alpha + \beta} \right) d\alpha d\beta \\
\leq f(x) \int_a^b \int_a^b \frac{\beta}{\alpha + \beta} d\alpha d\beta + f(y) \int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} d\alpha d\beta
\]
\[
= \frac{1}{2} (b - a)^2 \left[ f(y) + f(x) \right],
\]
which proves the last inequality in (19).

Let \((X, \| \cdot \|)\) be a normed linear space over the real or complex number fields. Then for any \( x, y \in X, p \geq 1 \) and \( b > a > 0 \) we have:

\[
\left\| \frac{x + y}{2} \right\|^p \\
\leq \frac{1}{(b - a)^2}
\times \int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} \left\{ \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} \right\}^p + \left\{ \frac{(2\beta + \alpha) x + ay}{2(\alpha + \beta)} \right\}^p \right\} d\alpha d\beta
\]
\[
\leq \int_0^1 \| (1 - t) x + ty \|^p dt
\]
\[
\leq \frac{1}{2} \left[ \frac{1}{(b - a)^2} \int_a^b \int_a^b \frac{\beta x + \alpha y}{\alpha + \beta} \left\| \frac{\beta x + \alpha y}{\alpha + \beta} \right\|^p d\alpha d\beta + \frac{\| y \|^p + \| x \|^p}{2} \right]
\]
\[
\leq \frac{\| y \|^p + \| x \|^p}{2}.
\]

The case of Breckner’s-convexity is as follows:

**Remark 3** Assume that the function \( f : C \subseteq X \rightarrow [0, \infty) \) is a Breckner \( s \)-convex function with \( s \in (0, 1) \). Let \( y, x \in C \) with \( y \neq x \) and assume that
the mapping \([0, 1] \ni t \mapsto f[(1-t)x + ty]\) is Lebesgue integrable on \([0, 1]\). Then for any \(b > a \geq 0\) we have

\[
\frac{2^{s-1}}{(b-a)^2} \times \int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} \left\{ f \left[ \frac{\alpha x + (2\beta + \alpha) y}{2(\alpha + \beta)} \right] + f \left[ \frac{(2\beta + \alpha) x + \alpha y}{2(\alpha + \beta)} \right] \right\} d\alpha d\beta
\]

\[
\leq \int_0^1 f[(1-t)x + ty] dt
\]

\[
\leq \frac{1}{s + 1} \left[ \frac{1}{(b-a)^2} \int_a^b \int_a^b f \left( \frac{\beta x + \alpha y}{\alpha + \beta} \right) d\alpha d\beta + f(y) + f(x) \right].
\]

We also have the norm inequalities:

\[
\frac{2^{s-1}}{(b-a)^2} \times \int_a^b \int_a^b \frac{\alpha}{\alpha + \beta} \left\{ \| \alpha x + (2\beta + \alpha) y \|_s^s + \| (2\beta + \alpha) x + \alpha y \|_s^s \right\} d\alpha d\beta
\]

\[
\leq \int_0^1 \| (1-t)x + ty \|_s^s dt
\]

\[
\leq \frac{1}{2} \left[ \frac{1}{(b-a)^2} \int_a^b \int_a^b \left\| \frac{\beta x + \alpha y}{\alpha + \beta} \right\|_s^s d\alpha d\beta + \| y \|_s^s + \| x \|_s^s \right],
\]

for any \(x, y \in X\), a normed linear space.

3 Inequalities for \(n\)-Points

In order to extend the above results for \(n\)-points, we need the following representation of the integral that is of interest in itself.

**Theorem 3** Let \(f : C \subseteq X \rightarrow \mathbb{C}\) be defined on the convex subset \(C\) of a real or complex linear space \(X\). Assume that for \(x, y \in C\) with \(x \neq y\) the mapping \([0, 1] \mapsto f((1-t)x + ty) \in \mathbb{C}\) is Lebesgue integrable on \([0, 1]\). Then for any partition

\[0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1\ with \ n \geq 1,
\]

we have the representation

\[
\int_0^1 f((1-t)x + ty) dt = \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \cdot
\]

\[
\cdot \int_0^1 f \{(1-u)[(1-\lambda_j)x + \lambda_j y] + u[(1-\lambda_{j+1})x + \lambda_{j+1}y]\} du.
\]
Proof. We have
\[
\int_0^1 f \left((1-t)x + ty\right) dt = \sum_{j=0}^{n-1} \int_{\lambda_j}^{\lambda_{j+1}} f \left((1-t)x + ty\right) dt. \tag{24}
\]

In the integral
\[
\int_{\lambda_j}^{\lambda_{j+1}} f \left((1-t)x + ty\right) dt, \quad j \in \{0, ..., n-1\},
\]
consider the change of variable
\[
u := \frac{1}{\lambda_{j+1} - \lambda_j} (t - \lambda_j), \quad t \in [\lambda_j, \lambda_{j+1}].
\]
Then
\[
d\nu = \frac{1}{\lambda_{j+1} - \lambda_j} dt,
\]
u = 0 for t = \lambda_j, u = 1 for t = \lambda_{j+1}, t = (1-u) \lambda_j + u \lambda_{j+1} and
\[
\int_{\lambda_j}^{\lambda_{j+1}} f \left((1-t)x + ty\right) dt = (\lambda_{j+1} - \lambda_j)
\times \int_0^1 f \left[(1 - (1-u) \lambda_j - u \lambda_{j+1}) x + ((1-u) \lambda_j + u \lambda_{j+1}) y\right] du
= (\lambda_{j+1} - \lambda_j)
\times \int_0^1 f \left[((1-u) (1 - \lambda_j) + u (1 - \lambda_{j+1})) x + ((1-u) \lambda_j + u \lambda_{j+1}) y\right] du
= \int_0^1 f \left\{((1-u) [(1-\lambda_j) x + \lambda_j y] + u [(1-\lambda_{j+1}) x + \lambda_{j+1} y]\right\} du
\]
for any j ∈ {0, ..., n-1}.

Making use of (24) and (25) we deduce the desired result (23). \qed

The following particular case is of interest and has been obtained in [17].

Corollary 3 In the the assumptions of Theorem 3 we have
\[
\int_0^1 f \left((1-t)x + ty\right) dt = \lambda \int_0^1 f \left\{(1-u)x + u [(1-\lambda)x + \lambda y]\right\} du \tag{26}
+ (1-\lambda) \int_0^1 f \left\{(1-u) [(1-\lambda)x + \lambda y] + uy\right\} du
\]
for any \lambda ∈ [0, 1].
Proof. Follows from (23) by choosing $0 = \lambda_0 \leq \lambda_1 = \lambda \leq \lambda_2 = 1$. □

The following result holds for $h$-convex functions:

**Theorem 4** Let $f : C \subseteq X \to \mathbb{C}$ be defined on the convex subset $C$ of a real or complex linear space $X$ and $f$ is $h$-convex on $C$ with $h \in L [0, 1]$. Assume that for $x, y \in C$ with $x \neq y$ the mapping $[0, 1] \mapsto f ((1 - t) x + ty) \in \mathbb{R}$ is Lebesgue integrable on $[0, 1]$. Then for any partition

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \geq 1,$$

we have the inequalities

$$\frac{1}{2h \left(\frac{1}{2}\right)} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) f \left\{ \left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\}$$

$$\leq \int_0^1 f ((1 - t) x + ty) dt$$

$$\leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[ f ((1 - \lambda_j) x + \lambda_j y) + f ((1 - \lambda_{j+1}) x + \lambda_{j+1} y) \right]$$

$$\times \int_0^1 h(u) du.$$  \hspace{1cm} (27)

Proof. Since $f$ is $h$-convex, then

$$f \left\{ (1 - u) [(1 - \lambda_j) x + \lambda_j y] + u [(1 - \lambda_{j+1}) x + \lambda_{j+1} y] \right\}$$

$$\leq h \left(1 - u\right) f ((1 - \lambda_j) x + \lambda_j y) + h \left(u\right) f ((1 - \lambda_{j+1}) x + \lambda_{j+1} y)$$

for any $u \in [0, 1]$ and for any $j \in \{0, ..., n-1\}$.

Integrating this inequality over $u \in [0, 1]$ we get

$$\int_0^1 f \left\{ (1 - u) [(1 - \lambda_j) x + \lambda_j y] + u [(1 - \lambda_{j+1}) x + \lambda_{j+1} y] \right\} du$$

$$\leq \int_0^1 \left\{ h \left(1 - u\right) f ((1 - \lambda_j) x + \lambda_j y) + h \left(u\right) f ((1 - \lambda_{j+1}) x + \lambda_{j+1} y) \right\} du$$

$$= f \left((1 - \lambda_j) x + \lambda_j y\right) \int_0^1 h \left(1 - u\right) du + f \left((1 - \lambda_{j+1}) x + \lambda_{j+1} y\right) \int_0^1 h \left(u\right) du$$

$$= [f \left((1 - \lambda_j) x + \lambda_j y\right) + f \left((1 - \lambda_{j+1}) x + \lambda_{j+1} y\right)] \int_0^1 h \left(u\right) du,$$

for any $j \in \{0, ..., n-1\}$.

Multiplying this inequality by $\lambda_{j+1} - \lambda_j \geq 0$ and summing over $j$ from 0 to $n-1$ we get, via the equality (23), the second inequality in (27).
Since $f$ is $h$-convex, then for any $v, w \in C$ we also have
\[ f(v) + f(w) \geq \frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{v + w}{2}\right). \]

If we write this inequality for
\[ v = (1 - u) \left[(1 - \lambda_j) x + \lambda_j y\right] + u \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \]
and
\[ w = u \left[(1 - \lambda_j) x + \lambda_j y\right] + (1 - u) \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \]
and take into account that
\[ \frac{v + w}{2} = \frac{1}{2} \left\{ \left[(1 - \lambda_j) x + \lambda_j y\right] + \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \]
\[ = \left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y, \]
then we get
\[ f\left\{ (1 - u) \left[(1 - \lambda_j) x + \lambda_j y\right] + u \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \]
\[ + f\left\{ u \left[(1 - \lambda_j) x + \lambda_j y\right] + (1 - u) \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \]
\[ \geq \frac{1}{h\left(\frac{1}{2}\right)} f\left\{ \left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y\right\} \]

for any $u \in [0, 1]$ and $j \in \{0, ..., n - 1\}$.

Integrating the inequality (28) over $u \in [0, 1]$ we get
\[ \int_0^1 f\left\{ (1 - u) \left[(1 - \lambda_j) x + \lambda_j y\right] + u \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \, du \]
\[ + \int_0^1 f\left\{ u \left[(1 - \lambda_j) x + \lambda_j y\right] + (1 - u) \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \, du \]
\[ \geq \frac{1}{h\left(\frac{1}{2}\right)} \int_0^1 f\left\{ \left(1 - \frac{\lambda_j + \lambda_{j+1}}{2}\right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y\right\} \, du \]

for any $j \in \{0, ..., n - 1\}$.

Since
\[ \int_0^1 f\left\{ (1 - u) \left[(1 - \lambda_j) x + \lambda_j y\right] + u \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \, du \]
\[ = \int_0^1 f\left\{ u \left[(1 - \lambda_j) x + \lambda_j y\right] + (1 - u) \left[(1 - \lambda_{j+1}) x + \lambda_{j+1} y\right] \right\} \, du, \]
then by (29) we get
\[
\int_0^1 f \left\{ (1-u) \left[ (1-\lambda_j) x + \lambda_j y \right] + u \left[ (1-\lambda_{j+1}) x + \lambda_{j+1} y \right] \right\} du \\
\geq \frac{1}{2h\left(\frac{1}{2}\right)} f \left\{ \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\}
\]
for any \( j \in \{0, ..., n-1\} \).

Multiplying this inequality by \( \lambda_{j+1} - \lambda_j \geq 0 \) and summing over \( j \) from 0 to \( n-1 \) we get, via the equality (23), the first inequality in (27). □

Remark 4 If we take \( 0 = \lambda_0 \leq \lambda_1 = \lambda \leq \lambda_2 = 1 \), then we get the first two inequalities in (14).

The case of convex functions is as follows:

Corollary 4 Let \( f : C \subseteq X \to \mathbb{R} \) be a convex function on the convex subset \( C \) of a real or complex linear space \( X \). Then for any partition
\[ 0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1 \] with \( n \geq 1 \),
and for any \( x, y \in C \) we have the inequalities
\[
f \left( \frac{x + y}{2} \right) \leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) f \left\{ \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\} \leq \int_0^1 f \left( (1-t) x + ty \right) dt \\
\leq \frac{1}{2} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[ f \left( (1-\lambda_j) x + \lambda_j y \right) + f \left( (1-\lambda_{j+1}) x + \lambda_{j+1} y \right) \right] \\
\leq \frac{f(x) + f(y)}{2}.
\]

Proof. The second and third inequalities in (30) follows from (27) by taking \( h(t) = t \).

By the Jensen discrete inequality
\[
\sum_{j=1}^m p_j f(z_j) \geq f \left( \sum_{j=1}^m p_j z_j \right),
\]
where $p_j \geq 0$, $j \in \{1, \ldots, m\}$ with $\sum_{j=1}^{m} p_j = 1$ and $z_j \in C$, $j \in \{1, \ldots, m\}$ we have

$$
\sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left\{ \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\}
\geq f \left\{ \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[ \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right] \right\}
= f \left\{ \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) - \frac{\sum_{j=0}^{n-1} (\lambda_{j+1}^2 - \lambda_j^2)}{2} \right\} x + \frac{\sum_{j=0}^{n-1} (\lambda_{j+1}^2 - \lambda_j^2)}{2} y
= f \left\{ \left( 1 - \frac{1}{2} \right) x + \frac{1}{2} y \right\} = f \left( \frac{x + y}{2} \right)
$$

and the first part of (30) is proved.

By the convexity of $f$ we also have

$$
\sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) [f ((1 - \lambda_j) x + \lambda_j y) + f ((1 - \lambda_{j+1}) x + \lambda_{j+1} y)]
\leq \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) [(1 - \lambda_j) f(x) + \lambda_j f(y) + (1 - \lambda_{j+1}) f(x) + \lambda_{j+1} f(y)]
= \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) [(2 - (\lambda_j + \lambda_{j+1})) f(x) + (\lambda_j + \lambda_{j+1}) f(y)]
= \left( 2 \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) - \sum_{j=0}^{n-1} (\lambda_{j+1}^2 - \lambda_j^2) \right) f(x) + \sum_{j=0}^{n-1} (\lambda_{j+1}^2 - \lambda_j^2) f(y)
= f(x) + f(y),
$$

which proves the last part of (30). □

**Remark 5** Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number fields. Then for any partition

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \geq 1,$$
and for any \( x, y \in X \) we have the inequalities
\[
\left\| \frac{x + y}{2} \right\|^p \leq \sum_{j=0}^{n-1} \left( \lambda_{j+1} - \lambda_j \right) \left\| \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\|^p
\]
\[
\leq \int_0^1 \left\| (1 - t) x + ty \right\|^p dt
\]
\[
\leq \frac{1}{2} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[ \left\| (1 - \lambda_j) x + \lambda_j y \right\|^p + \left\| (1 - \lambda_{j+1}) x + \lambda_{j+1} y \right\|^p \right]
\]
\[
\leq \frac{\left\| x \right\|^p + \left\| y \right\|^p}{2},
\]
where \( p \geq 1 \).

**Corollary 5** Let \( f : C \subseteq X \to \mathbb{R} \) be defined on a convex subset \( C \) of a real or complex linear space \( X \) and \( f \) is Breckner \( s \)-convex on \( C \) with \( s \in (0, 1) \). Assume that for \( x, y \in C \) with \( x \neq y \) the mapping \([0, 1] \mapsto f ((1 - t) x + ty) \in \mathbb{R} \) is Lebesgue integrable on \([0, 1] \). Then for any partition
\[
0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \geq 1,
\]
we have the inequalities
\[
2^{s-1} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) f \left\{ \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\} \leq \int_0^1 f ((1 - t) x + ty) dt
\]
\[
\leq \frac{1}{s + 1} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[ \left\| (1 - \lambda_j) x + \lambda_j y \right\|^s + \left\| (1 - \lambda_{j+1}) x + \lambda_{j+1} y \right\|^s \right].
\]

Since, for \( s \in (0, 1) \), the function \( f (x) = \| x \|^s \) is Breckner \( s \)-convex on the normed linear space \( X \), then by (32) we get for any \( x, y \in X \)
\[
2^{s-1} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left\| \left( 1 - \frac{\lambda_j + \lambda_{j+1}}{2} \right) x + \frac{\lambda_j + \lambda_{j+1}}{2} y \right\|^s \leq \int_0^1 \left\| (1 - t) x + ty \right\|^s dt
\]
\[
\leq \frac{1}{s + 1} \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \left[ \left\| (1 - \lambda_j) x + \lambda_j y \right\|^s + \left\| (1 - \lambda_{j+1}) x + \lambda_{j+1} y \right\|^s \right].
\]
References


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