Note on Matuzsewska-Orlicz indices and Zygmund inequalities

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Abstract

In this note we call attention to the fact that there exist some relations between the Matuszewska-Orlicz indices \( m(\varphi) \) and \( M(\varphi) \) of the function \( \varphi \), and possible values of the constants in Zygmund type inequalities.

Key Words: Matuszewska-Orlicz indices, Zygmund type inequalities, almost monotonic functions

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1 Introduction

The main goal of this note is to call attention to the fact that there exist some relations between the Matuszewska-Orlicz indices \( m(\varphi) \) and \( M(\varphi) \) of the function \( \varphi \), and possible values of the constants \( c_\varphi \) and \( C_\varphi \) in the inequalities

\[
\int_0^h \frac{\varphi(t)}{t} dt \leq \frac{1}{c_\varphi} \varphi(h), \tag{1}
\]

\[
\int_h^\ell \frac{\varphi(t)}{t} dt \leq \frac{1}{C_\varphi} \varphi(h), \tag{2}
\]

where \( 0 < h \leq \ell < \infty \), \( \varphi \) is a non-negative function, see Theorems 3.1 and 4.1.
Inequalities (1) and (2) are known as Zygmund type inequalities, we refer for instance to [1], where under some monotonicity conditions on \( \varphi \) there was shown in particular that Zygmund inequalities are equivalent to the so called Lozinsky and Bary-Stechkin conditions. In [2], [7] it was shown that monotonicity conditions on \( \varphi \) may be replaced by that of almost monotonicity, or more generally, by the condition \( \varphi \in \tilde{W} \), see Definition 2.1; recall that a non-negative function \( \varphi \) is called almost increasing if there exists a constant \( c \geq 1 \) such that \( \varphi(x) \leq \varphi(y) \) for all \( x \leq y \).

Note that we prefer to write constants on the right-hand sides of (1)-(2) as \( \frac{1}{c} \) and \( \frac{1}{C} \) by reasons which become clear in the sequel, see for instance Lemma 3.1 and inequality (5).

2 Preliminaries

The Matuszewska-Orlicz indices known in the theory of Orlicz spaces (see [5], [3] and [4], where they were studied mainly for Young functions \( \varphi \)), are defined as

\[
m(\varphi) = \sup_{t > 1} \frac{\ln \left( \lim_{h \to 0} \frac{\varphi(th)}{\varphi(h)} \right)}{\ln t} = \lim_{t \to 0} \frac{\ln \left( \lim_{h \to 0} \frac{\varphi(th)}{\varphi(h)} \right)}{\ln t},
\]

\[
M(\varphi) = \inf_{t > 1} \frac{\ln \left( \lim_{h \to 0} \frac{\varphi(th)}{\varphi(h)} \right)}{\ln t} = \lim_{t \to \infty} \frac{\ln \left( \lim_{h \to 0} \frac{\varphi(th)}{\varphi(h)} \right)}{\ln t},
\]

the definition being applicable to any non-negative-function \( \varphi \), and

\[-\infty \leq m(\varphi) \leq M(\varphi) \leq +\infty\]

in this case.

Note that for \( \varphi_\gamma(t) = t^\gamma \varphi(t) \) we have

\[m(\varphi_\gamma) = \gamma + m(\varphi) \quad \text{and} \quad M(\varphi_\gamma) = \gamma + M(\varphi).\]

**Definition 2.1.** By \( W = W([0, \ell]) \) we denote the class of non-negative almost increasing functions on \([0, \ell]\), positive on \((0, \ell)\) and by \( \tilde{W} = \tilde{W}([0, \ell]) \) we denote the class of functions on \([0, \ell]\), such that there exists an \( a \in \mathbb{R}^1 \) such that the function \( x^a \varphi(x) \in W \).

In the case \( \varphi \in \tilde{W} \), one has

\[-\infty < m(\varphi) \leq M(\varphi) \leq +\infty.\]

Various properties of the indices \( m(\varphi) \) and \( M(\varphi) \) were obtained in [3] and [4], and in [2], [7], [8], [9], [10], [11], [12] in connection with study of various operators in generalized H"older spaces, where in particular it was shown that the validity of the Zygmund inequalities for a function \( \varphi(t) \) may be characterized in terms of the indices \( m(\varphi), M(\varphi) \).
In particular, the following property is known (for the proof see [2], Theorems 3.1 and 3.2 for \( \varphi \in \mathring{W} \), as stated in Theorem 2.1, and [3], Thm 6.4 or [4], Thm 11.8 under a different definition of the indices and other assumptions on \( \varphi \))

**Theorem 2.1.** Let \( \varphi \in \mathring{W} \). Then

\[
\int_0^h \varphi(t) t^{1+\gamma} dt \leq c \frac{\varphi(h)}{h^\gamma} \iff \gamma < m(\varphi),
\]

(3)

\[
\int_h^t \varphi(t) t^{1+\nu} dt \leq c \frac{\varphi(h)}{h^\nu} \iff \nu > M(\varphi).
\]

(4)

3. **A relation between the index \( m(\varphi) \) and the constant \( c_\varphi \)**

Given a non-negative function \( \varphi \), let

\[
I_-(\varphi) = \left\{ \gamma \in \mathbb{R}^1 : \text{there exists } c = c(\varphi, \gamma) \text{ such that } \int_0^h \frac{\varphi(t)}{t^{1+\gamma}} dt \leq \frac{1}{c} \frac{\varphi(h)}{h^\gamma} \right\}.
\]

Obviously, if \( \gamma \in I_-(\varphi) \), then \( \gamma - \alpha \in I_-(\varphi) \) for any \( \alpha > 0 \), so that \( I_-(\varphi) \) may be only an infinite interval starting from \(-\infty\). For functions \( \varphi \in \mathring{W} \) it is known that the set \( I_-(\varphi) \) is an open interval with the exactly calculated upper bound:

\[
I_-(\varphi) = (-\infty, m(\varphi)),
\]

(1)

which follows from [3].

In Lemma 3.1 we show that the fact itself that this interval is open, is valid for an arbitrary non-negative function \( \varphi \), without any assumption on almost monotonicity of \( \varphi \), and find a relation between the constants \( c(\varphi, \gamma) \) and \( c(\varphi, \gamma + \varepsilon) \).

**Lemma 3.1.** Let \( \varphi(t) \) be a non-negative function on \([0, \ell]\) such that the integral \( \int_0^t \frac{\varphi(s)}{s} ds \) exists for every \( t \in (0, \ell) \). If there holds inequality [1] with some \( c_\varphi > 0 \), then for any \( \varepsilon \in (0, c_\varphi) \) there also holds the inequality

\[
\int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}} dt \leq \frac{1}{c_\varphi - \varepsilon} \frac{\varphi(h)}{h^\varepsilon}
\]

(2)

where \( c \) is the same as in [1].

**Proof.** Let

\[
\Phi(t) = \int_0^t \frac{\varphi(s)}{s} ds.
\]
The formula is valid
\[ \int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}} dt = \frac{\Phi(h)}{h^\varepsilon} + \varepsilon \int_0^h \frac{\Phi(t)}{t^{1+\varepsilon}} dt. \tag{3} \]

Indeed,
\[ \varepsilon \int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}} dt = \varepsilon \int_0^h \frac{dt}{t^{1+\varepsilon}} \int_0^t \frac{\varphi(s)}{s} ds = \int_0^h \frac{\varphi(s)}{s} \left( \frac{1}{s^\varepsilon} - \frac{1}{h^\varepsilon} \right) ds \]
which yields (3).

Since \( \Phi(h) \leq \frac{1}{c_\varphi} \varphi(h) \) by (1), from (3) we obtain
\[ \int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}} dt \leq \frac{\varphi(h)}{c_\varphi h^\varepsilon} + \varepsilon \int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}} dt, \]
from which (2) follows. \( \square \)

**Corollary 3.1.** Let \( \varphi \) be a non-negative function on \([0, \ell]\) such that \( \int_0^\ell \frac{\varphi(t)}{t^{1+\gamma}} dt \) exists, \( \gamma \in \mathbb{R}^1 \).

Then
\[ \int_0^h \frac{\varphi(t)}{t^{1+\gamma}} dt \leq \frac{1}{c_\gamma} \frac{\varphi(h)}{h^\gamma} \quad \Rightarrow \quad \int_0^h \frac{\varphi(t)}{t^{1+\gamma+\varepsilon}} dt \leq \frac{1}{c_\gamma - \varepsilon} \frac{\varphi(h)}{h^{\gamma+\varepsilon}} \]
for any \( \varepsilon < c_\gamma \).

**Remark 3.1.** In case we pass from the factor \( \frac{1}{t} \) in (2) to a power of the logarithmic function, the corresponding statement becomes
\[ \int_0^h \frac{\varphi(t)}{t} dt \leq \frac{1}{c_\varphi} \varphi(h) \quad \Rightarrow \quad \int_0^h \frac{\varphi(t)}{t} \left( \frac{\ln h}{t} \right)^n dt \leq \frac{1}{c_\varphi n!} \varphi(h), \tag{4} \]
where \( n = 1, 2, 3, ... \) which may be obtained by the successive application of the given inequality:
\[ \varphi(h) \geq c_\varphi \int_0^h \frac{\varphi(t)}{t} dt \geq c_\varphi^2 \int_0^h \frac{dt}{t} \int_0^t \frac{\varphi(s)}{s} ds = c_\varphi^2 \int_0^h \frac{\varphi(s)}{s} \ln \frac{h}{s} ds \quad \text{etc.} \]

**Theorem 3.1.** Let \( \varphi \in \widetilde{W} \). If there holds inequality (1) with some constant \( c_\varphi > 0 \), then
\[ c_\varphi \leq m(\varphi). \tag{5} \]
Proof. Suppose to the contrary that \( m(\varphi) < c_\varphi \). By Lemma 3.1, inequality (2) holds with every \( \varepsilon \in (0, c_\varphi) \), in particular, with every \( \varepsilon \in (\lambda, c_\varphi) \), \( \lambda = \max\{m(\varphi), 0\} \), which is impossible, because for \( \varphi \in \widehat{W} \), inequality (2) implies \( m(\varphi) > \varepsilon \) by 3.

Corollary 3.2. For the index \( m(\varphi) \) of a function \( \varphi \in W \) the estimate holds

\[
m(\varphi) \geq \inf_{t>0} \frac{\varphi(t)}{\Phi(t)} = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)},
\]

where \( \Phi(t) = \int_0^t \frac{\varphi(s)}{s} ds \).

Proof. Let \( A = \sup_{h>0} \frac{\Phi(h)}{\varphi(h)} \). Let first \( A = \infty \). Then the right-hand side of (6) is zero and also \( m(\varphi) = 0 \). Indeed, we have \( m(\varphi) \geq 0 \) for \( \varphi \in W \) and in case \( m(\varphi) > 0 \) there holds (1) with a finite constant \( c_\varphi \), which would mean that \( A < \infty \). Therefore, (6) trivially holds in the case \( A = \infty \).

Let \( A < \infty \). Then (1) obviously holds with \( c_\varphi = \frac{1}{A} \). Then \( \frac{1}{A} \leq m(\varphi) \) by Lemma 3.1 which is inequality (6).

Remark 3.2. In case of power functions \( \varphi(t) = t^\lambda \) we have

\[
m(\varphi) = M(\varphi) = \inf_{t>0} \frac{\varphi(t)}{\Phi(t)} = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} = \lambda,
\]

but in the general case it may be that \( m(\varphi) > \inf_{t>0} \frac{\varphi(t)}{\Phi(t)} \).

4 A relation between the index \( M(\varphi) \) and the constant \( C_\varphi \)

Similarly to the previous section we reveal a relation between the upper index \( M(\varphi) \) and the constant \( C_\varphi \) in the Zygmund inequality (2).

Let

\[
I_+(\varphi) = \left\{ \gamma \in \mathbb{R}^1 : \text{there exists } C = C(\varphi, \gamma) \text{ such that } \int_h^l \frac{\varphi(t)}{t^{1+\gamma}} dt \leq \frac{1}{C} \frac{\varphi(h)}{h^\gamma} \right\}.
\]

For functions \( \varphi \in \widehat{W} \) it is known that

\[
I_+ = (M(\varphi), +\infty),
\]

see (4). The following lemma exactifies the statement on the openness of the interval \( (M(\varphi), +\infty) \) for an arbitrary non-negative function.
**Lemma 4.1.** Let \( \varphi(t) \) be a non-negative function on \([0, \ell]\) such that the integral \( \int_0^\ell \frac{\varphi(s)}{s} ds \) exists for every \( t \in (0, \ell) \). If there holds inequality (2) with some \( C_\varphi > 0 \), then for any \( \varepsilon \in (0, C_\varphi) \) there also holds the inequality

\[
\int_0^\ell \frac{\varphi(t)}{t^{1-\varepsilon}} dt \leq \frac{1}{C_\varphi - \varepsilon} h^\varepsilon \varphi(h)
\]

(1)

where \( C_\varphi \) is the same as in (2).

**Proof.** Lemma 4.1 was proved in [6]. We give the proof here for the completeness of presentation. Let \( \Phi_1(t) = \int_0^t \frac{\varphi(s)}{s} ds \). Similarly to (3) we have

\[
\int_0^\ell \frac{\varphi(t)}{t^{1-\varepsilon}} dt = h^\varepsilon \Phi_1(h) + \varepsilon \int_0^\ell \frac{\Phi_1(t)}{t^{1-\varepsilon}} dt
\]

(2)

by direct verification. Since \( \Phi_1(h) \leq \frac{1}{C_\varphi} \varphi(h) \) by (2), from (2) we obtain

\[
C_\varphi \int_0^\ell \frac{\varphi(t)}{t^{1-\varepsilon}} dt \leq h^\varepsilon \varphi(h) + \varepsilon \int_0^\ell \frac{\varphi(t)}{t^{1-\varepsilon}} dt,
\]

from which (1) follows. \( \square \)

**Lemma 4.2.** Let \( \varphi \in \widetilde{W} \). If there holds inequality (2) with some constant \( C_\varphi > 0 \), then

\[
M(\varphi) \leq -C_\varphi.
\]

**Proof.** Suppose to the contrary that \( M(\varphi) > -C_\varphi \). By Lemma 4.1 inequality (1) holds with every \( \varepsilon \in (0, C_\varphi) \), in particular, with every \( \varepsilon \in (\mu, C_\varphi) \), \( \mu = \max\{-M(\varphi), 0\} \), which is impossible, because for \( \varphi \in \widetilde{W} \), inequality (1) implies \( M(\varphi) < -\varepsilon \) by (4). \( \square \)

**Theorem 4.1.** If a function \( \varphi \in \widetilde{W} \) admits estimate (2) with some constant \( C_\varphi > 0 \), then for the index \( M(\varphi) \) the estimate holds

\[
M(\varphi) \leq -\inf_{0 < t \leq \ell} \frac{\varphi(t)}{\Phi_1(t)} = \sup_{0 < t \leq \ell} t \frac{\Phi_1'(t)}{\Phi_1(t)},
\]

(3)

where \( \Phi_1(t) = \int_0^t \frac{\varphi(s)}{s} ds \).

**Proof.** Let \( A_1 = \sup_{0 < t \leq \ell} \frac{\Phi_1(t)}{\varphi(t)} \). Inequality (2) obviously holds with \( C_\varphi = \frac{1}{A_1} \). Then \( \frac{1}{A_1} \leq -M(\varphi) \) by Lemma 4.2, which is inequality (3). \( \square \)
Remark 4.1. The indices

\[ p(\varphi) = \inf_{0 < x \leq \ell} \frac{x \varphi'(x)}{\varphi(x)}, \quad q(\varphi) = \sup_{0 < x \leq \ell} \frac{x \varphi'(x)}{\varphi(x)} \] (4)

are known as Simonenko indices, see [13], and it is known that

\[ p(\varphi) \leq m(\varphi) \leq M(\varphi) \leq q(\varphi), \] (5)

see [4], Theorem 11.11. In these terms, inequalities (6) and (3), in case \( \varphi \in W \), mean that

\[ p(\Phi) \leq m(\varphi) \leq M(\varphi) \leq q(\Phi_1). \] (6)

Observe that although we can write, for instance,

\[ p(\Phi) \leq m(\Phi) \leq M(\Phi) \leq q(\Phi), \]

to derive the left-hand side inequality \( p(\Phi) \leq m(\varphi) \) in (6) from here, we would like to have the property \( m(\Phi) = m(\varphi) \), which is true in the case \( 0 < m(\varphi) \leq M(\varphi) < \infty \) because \( \Phi \sim \varphi \) in this case and then the functions \( \Phi \) and \( \varphi \) have coinciding indices, see [4], Theorem 11.4. Similarly one has \( M(\Phi_1) = M(\varphi) \) when \( -\infty < m(\varphi) \leq M(\varphi) < 0 \).

5 A generalization of Lemmas 3.1 and 4.1

Based on the passage from (1) to (2) and the example given in (4), we now consider a possibility to trace a similar passage when one deals with the scale of functions more fine than just the scale of power (or power-logarithmic) functions.

In the sequel the notation \( AC(0, \ell) \) stands for the set of functions on \( (0, \ell) \) absolutely continuous on every closed subinterval of \( (0, \ell) \).

Lemma 5.1. Suppose that

\[ \int_0^h \frac{\varphi(t)}{t} dt \leq \frac{1}{c_0} \varphi(h) \] (1)

for some \( c_0 > 0 \). Then a similar inequality

\[ \int_0^h \frac{\varphi(t)}{t \nu(t)} dt \leq \frac{1}{c_0 - \delta} \frac{\varphi(h)}{\nu(h)} \] (2)

holds, where \( \nu(t) \) is any non-negative function on \( [0, \ell] \) such that \( \frac{1}{\nu} \in AC(0, \ell) \), and

\[ \delta =: \sup_{t \in [0, \ell]} \frac{t |\nu'(t)|}{\nu(t)} < c_0. \] (3)
Proof. Integration by parts yields
\[ \int_0^h \frac{\varphi(t) dt}{t \nu(t)} = \frac{\Phi(h)}{\nu(h)} + \int_0^h \frac{\nu'(t)}{\nu^2(t)} \Phi(t) dt \] (4)
since \( \lim_{h \to 0} \frac{\Phi(h)}{\nu(h)} = 0 \). To check the latter, in view of (1) it suffices to show that \( \lim_{h \to 0} \frac{\varphi(h)}{\nu(h)} = 0 \), for which it is sufficient to verify that \( m \left( \frac{x}{h} \right) > 0 \). Since \( m \left( \frac{x}{h} \right) \geq m(\varphi) + m \left( \frac{1}{\nu} \right) = m(\varphi) - M(\nu) \), we then may only check that \( M(\nu) < m(\varphi) \). The latter follows from condition (3), which implies that \( M(\nu) \leq q(\nu) < c_0(\leq m(\varphi)) \).

From (4), by assumption (1) we obtain
\[ \int_0^h \frac{\varphi(t) dt}{t \nu(t)} \leq \frac{1}{c_0} \left[ \frac{\varphi(h)}{\nu(h)} + \int_0^h \frac{\nu'(t)}{\nu^2(t)} \varphi(t) dt \right] \] (5)
or
\[ \int_0^h \left( 1 - \frac{1}{c_0} \frac{t|\nu'(t)|}{|\nu(t)|} \right) \varphi(t) \frac{dt}{t \nu(t)} \leq \frac{1}{c_0} \frac{\varphi(h)}{\nu(h)}. \] (6)

By assumption (3) we have \( 1 - \frac{1}{c_0} \frac{t|\nu'(t)|}{|\nu(t)|} \geq 1 - \frac{\delta}{c_0} \), which yields (2).

Lemma 5.2. Suppose that
\[ \int_0^\ell \frac{\varphi(t)}{t} dt \leq \frac{\varphi(h)}{C_0} \] (7)
for some \( C_0 > 0 \). Then a similar inequality
\[ \int_0^\ell \frac{\varphi(t) \lambda(t)}{t} dt \leq \frac{\lambda(h) \varphi(h)}{C_0 - \delta} \] (8)
holds, where \( \lambda(t) \) is any non-negative function in \( AC(0, \ell) \), and
\[ \delta =: \sup_{t \in [0, \ell]} \frac{t|\lambda'(t)|}{\lambda(t)} < C_0. \] (9)

Proof. Integrating by parts, we obtain
\[ \int_0^\ell \lambda(t) \frac{\varphi(t)}{t} dt = \lambda(h) \Phi_1(h) + \int_0^\ell \lambda'(t) \Phi_1(t) dt, \quad \Phi_1(t) = \int_0^t \frac{\varphi(s)}{s} ds. \] (10)

By assumption (7) we then have
\[ \int_0^\ell \frac{\lambda(t) \varphi(t)}{t} dt \leq \frac{1}{C_0} \left[ \lambda(h) \varphi(h) + \int_0^\ell \frac{\lambda'(t)}{t} \varphi(t) dt \right] \] (11)
or
\[ \int_0^\ell \left( 1 - \frac{1}{C_0} \frac{t|\lambda'(t)|}{\lambda(t)} \right) \frac{\lambda(t) \varphi(t)}{t} dt \leq \frac{1}{C_0} \lambda(h) \varphi(h), \] (12)
which yields (8) by (9).
References


