On injectors of finite groups

Shitian Liu* and Runshi Zhang**

* School of Science, Sichuan University of Science & Engineering, Zigong 643000 PR.China.
liust@suse.edu.cn

** School of Science, Sichuan University of Science & Engineering Zigong 643000 PR.China.
zhangrs-75@163.com

Received by the editors May 07, 2010; accepted for publication February 15, 2010.

Abstract

If $F$ is a non-empty Fitting class, $\pi = \pi(F)$ and $G$ a group such that every chief factor of $G/G_F$ is an $C^\pi_x$-group. Then $G$ has at least one $F$-injector. This result is used to resolve an open problem and generalize some known results.

Key Words: Fitting class, Soluble, Chief factor, $F$-injector.
Mathematics Subject Classification 2000: 20D10, 20F17

1 Introduction

All groups in this paper are finite. Fischer, Gaschütz, and Hartley [1] proved that for any Fitting class $F$ and any finite solvable group $G$ there exist $F$-injectors and any two of them are conjugate in $G$. A class $F$ of groups is a Fitting class if (i) $G \in F$, $N \triangleleft G$ implies that $N \in F$ and (ii) $N_1, N_2 \triangleleft G$, $N_1, N_2 \in F$ implies that $N_1N_2 \in F$. Fitting classes were introduced by Fischer, Gaschütz, and Hartley [1]. If $F$ is a Fitting class, each group $G$ possesses a unique maximal normal $F$-subgroup called the $F$-radical of $G$ and denoted by $G_F$, which contains each subnormal $F$-subgroup of $G$. Furthermore if $N$ is subnormal in $G$, then $N_F = N \cap G_F$. A subgroup $V$ of $G$ is called an $F$-injector of $G$ if $V \cap N$ is $F$-maximal in $N$ for each subnormal subgroup $N$ of $G$. In particular, an $F$-injector $V$ of $G$ lies in $F$ and contains $G_F$. Tomkinson in [2] proved that let $F$ be a Fitting class, then the $G$-group $G$ possesses $F$-injectors, where $G$ is the class of
periodic locally soluble FC-groups. Flavell in [3] proved that if \( G \) is a group whose local subgroups are \( N \)-constrained, then all nilpotent injectors of \( G \) are conjugate. Guo, and Vorob’ev [4] described the \( \mathcal{H} \)-injectors associated with a Hartley class \( \mathcal{H} \). And some good results are given by some authors (see [5,6,7,8]).

In this note, we give the following notations (see [9, p386-387]):

\( \pi \): any set of primes.

\( E_\pi : G \) has at least one Hall \( \pi \)-subgroup;

\( C_\pi : G \) satisfies \( E_\pi \) and any two Hall \( \pi \)-subgroups of \( G \) are conjugate in \( G \).

\( E_\pi^n : G \) has a nilpotent Hall \( \pi \)-subgroup.

\( C_\pi^* : G \) satisfies \( C_\pi \) and its Hall \( \pi \)-subgroup are soluble.

Guo, and Li [10] gave that, let \( \mathcal{F} \) be a non-empty Fitting class, \( \pi = \pi(G) \) and \( G \) a group such that every chief factor of \( G/G_{\mathcal{F}} \) is an \( E_\pi^n \)-group, then \( G \) has at most one \( \mathcal{F} \)-injector and any two \( \mathcal{F} \)-injectors are conjugate in \( G \).

Concerning Fitting classes and \( \mathcal{F} \)-injectors, the following problem arose:

**Problem** (see [11]). Let \( \mathcal{F} \) is a local Fitting class. Could we describe the \( \mathcal{F} \)-injectors of a group?

In this note, we will partially deal with the problem and prove the following main theorem.

**Theorem 1.1.** Let \( \mathcal{F} \) be a non-empty Fitting class, \( \pi = \pi(\mathcal{F}) \) and \( G \) a group such that every chief factor of \( G/G_{\mathcal{F}} \) is an \( C_\pi^* \)-group. Then

1. \( G \) has at least one \( \mathcal{F} \)-injector.
2. Any two \( \mathcal{F} \)-injector are exactly all the \( \mathcal{F} \)-maximal subgroups which contain the \( \mathcal{F} \)-radical \( G_{\mathcal{F}} \).
3. In any \( G \) there exist \( \mathcal{F} \)-injectors and any two of them are conjugate in \( G \).

For some notion and notations, the reader is referred to Ballester-Bolinches and Ezquerro [12], and Doerk and Hawkes [13].

# Preliminaries

**Definition 2.1** ([11]). A class \( \mathcal{F} \) of groups is a Fitting class if

(i) \( G \in \mathcal{F} \), \( N \triangleleft G \) implies that \( N \in \mathcal{F} \) and

(ii) \( N_1, N_2 \triangleleft G \), \( N_1, N_2 \in \mathcal{F} \) implies that \( N_1N_2 \in \mathcal{F} \).

**Definition 2.2** (see [12] or [13]). A subgroup \( V \) of \( G \) is called an \( \mathcal{F} \)-injector of \( G \) if \( V \cap N \) is \( \mathcal{F} \)-maximal in \( N \) for each subnormal subgroup \( N \) of \( G \).

**Lemma 2.1** ([11]). Let \( \mathcal{F} \) be a Fitting class. Then a soluble group \( G \) has at most one \( \mathcal{F} \)-injector and any two \( \mathcal{F} \)-injectors of \( G \) are conjugate in \( G \).
Lemma 2.2 ([10]). Let $\mathcal{F}$ be a Fitting class, and $H$ an $\mathcal{F}$-injector of $G$. Then the following statements hold:

1. $H$ is an $\mathcal{F}$-maximal subgroup of $G$;
2. $G_{\mathcal{F}} \leq H$;
3. For every $x \in H$, $H^x$ is also an $\mathcal{F}$-injector of $G$;
4. If $K$ is subnormal subgroup of $G$, then $H \cap K$ is an $\mathcal{F}$-injector of $K$.

Lemma 2.3 ([1]). If $V$ is an $\mathcal{F}$-injector of $G$ and $V \leq H \leq G$, then $V$ is an $\mathcal{F}$-injector of $H$.

Lemma 2.4 ([1]). Let $N \trianglelefteq G$ with $G/N$ nilpotent. If $V_1$ and $V_2$ are $\mathcal{F}$-maximal in $G$ and $V_1 \cap N = V_2 \cap N$ is $\mathcal{F}$-maximal in $N$, then $V_1$ and $V_2$ are conjugate in $G$.

Lemma 2.5 ([15, p334], or [14, Corollary 7.3.12]). Let $\mathcal{F}$ be a Fitting class, if $H$ is subnormal subgroup of $G$, then $H_{\mathcal{F}} = H \cap G_{\mathcal{F}}$.

Lemma 2.6 ([13, IX-Lemma 1.6]). Let $\mathcal{F}$ be a Fitting class, and let $G$ be a finite soluble group. Let $N \trianglelefteq G$, and let $L$ be a subgroup of $G$ such that $L \cap N$ is an $\mathcal{F}$-injector of $N$. Assume that either

1. $G/N$ is nilpotent, and $L$ is $\mathcal{F}$-maximal in $G$, or
2. $L \in \mathcal{F}$ and $LN = G$.

Then $L$ is an $\mathcal{F}$-injector of $G$.

Lemma 2.7 ([9, Theorem C1]). If $G$ has a series in which every factor is a $C_{\pi}^s$-group, then $G$ is a $C_{\pi}^s$-group and every Hall $\pi$-subgroup of $G$ is solvable.

Lemma 2.8. Let $\mathcal{F}$ be a Fitting class and $G$ a group. Suppose that $G/G_{\mathcal{F}}$ is soluble and $G/N$ is soluble. If $V$ is an $\mathcal{F}$-maximal subgroup of $G$ and $V \cap N$ is an $\mathcal{F}$-injector of $N$, then $V$ is an $\mathcal{F}$-injector of $G$.

Proof. Assume that the Lemma is not true and $G$ is a minimal-order-counter-example. By [12, Theorem 2.4.27], $G$ has a unique conjugate class of $\mathcal{F}$-injectors. Let $V_0$ be a $\mathcal{F}$-injector of $G$, then, by Lemma 2.2(1), $V_0$ is a maximal $\mathcal{F}$-subgroup of $G$.

Cases 1. $NV < G$.

Since $G/N$ is soluble, $VN/N$ is soluble. Obviously $V$ is also a maximal $\mathcal{F}$-subgroup of $VN$. By Lemma 2.5, $(VN)_{\mathcal{F}} = NV \cap G_{\mathcal{F}}$. Hence the quotient $NV/(NV)_{\mathcal{F}} = NV/(NV \cap G_{\mathcal{F}}) \cong NVG_{\mathcal{F}}/G_{\mathcal{F}} \leq G/G_{\mathcal{F}}$ is soluble. Thus, the minimal choice of $G$ implies that $V$ is an $\mathcal{F}$-injector of $G$. By [12, Theorem 2.4.27], $G$ has a unique conjugate class of $\mathcal{F}$-injectors, there exist an element $x \in NV$ such that $(V_0 \cap NV)^x = V$, and so $V \leq V_0^x$. Since an $\mathcal{F}$-maximal subgroup of $G$, $V = V_0^x$, and so $V$ is an $\mathcal{F}$-injector of $G$ by virtue of Lemma 2.2(3), a contradiction.

Cases 2. $NV = G$.
Let $M$ be a maximal normal subgroup of $G$ containing $N$. Since $G/N$ is soluble, $M/N$ is soluble. It is easy to see that $V \cap M \triangleleft V$. Let $V_1$ be a maximal $\mathcal{F}$-subgroup of $M$ with $V \cap N \leq V_1$. Since $V \cap N = (V \cap M) \cap N \leq V_1 \cap N$ and $V_1 \cap N = (V_1 \cap N) \cap (NV) = (V \cap N) \cap (V_1 \cap N) \leq V \cap N$, $V_1 \cap N = V \cap N$ is an $\mathcal{F}$-injector of $N$ by hypotheses, and, by Lemma 2.2(3), the quotient $M/M_{\mathcal{F}} = M/(M \cap G_{\mathcal{F}}) \cong MG_{\mathcal{F}}/G_{\mathcal{F}} \leq G/G_{\mathcal{F}}$ is soluble. The minimal choice of $G$ implies that $V_1$ is an $\mathcal{F}$-injector of $M$. Since $M = (NV) \cap M = N(V \cap M) \leq NV_1 \leq G$, $NV_1 = G$ or $M$. If the former, then, $G/N = NV/N \cong V/V \cap N \cong V_1N/N \cong V_1V_1 \cap N$. Comparing the order, we have $V_1 = V^x$ for some $x \in G$. By Lemma 2.2(3), $V_1$ is an $\mathcal{F}$-injector of $G$, a contradiction. So have $M = NV_1$, and $|N(V \cap M)| = |NV_1| = |M|$. This shows $|N||V \cap M|/|V \cap N| = |N||V_1|/|N \cap V_1|$ and hence $(V \cap M)^x = V_1$ for some $x \in G$. By Lemma 2.2(3), $V \cap M$ is an $\mathcal{F}$-injector of $M$. On the other hand, by Lemma 2.2(4), $V_0 \cap M$ is an $\mathcal{F}$-injector of $M$. By [12, Theorem 2.4.27], there exists an $x \in M$ such that $V_0 \cap M = (V \cap M)^x = V^x \cap M$. Moreover, by Lemma 2.2(1), $V_0, V^x$ are $\mathcal{F}$-maximal subgroup of $M$ and $G_{\mathcal{F}} \leq V \cap V_0$. By [16, Lemma 2.3], $V, V_0$ are conjugate in $G$, and so $V$ is an $\mathcal{F}$-injector of $G$, a contradiction.

This completes the proof. □

Lemma 2.9. Let $\mathcal{F}$ be non-empty Fitting class and $G$ a group. If every chief factor of $G/G_{\mathcal{F}}$ is an $C_{n^*}$-group and $N \triangleleft G$, then every chief factor of $N/N_{\mathcal{F}}$ is also an $C_{n^*}$-group.

Proof. By hypotheses, there exists a series

$$G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$$

Such that every chief factor are $C_{n^*}$-group. Since $N_{\mathcal{F}} = N \cap G_{\mathcal{F}}$ by Lemma 2.5, $N/N_{\mathcal{F}} = N/(N \cap G_{\mathcal{F}}) \cong NG_{\mathcal{F}}/G_{\mathcal{F}} \leq G/G_{\mathcal{F}}$. It follows that, intersection of $N$ and the above series is the series such that every chief factor of $N/N_{\mathcal{F}}$ is also an $C_{n^*}$-group.

This completes the proof. □

3 Some results

In this section, we will give the proof of the main theorem 1.1 and some applications.

The proof of Theorem 1.1

Proof. Our proof proceeds via a number of steps.

Step 1. $G$ is a $C_\pi$-group and if $H$ is a Hall $\pi$-subgroup of $G$, then $H/H_{\mathcal{F}}$ is soluble.

By Lemma 2.7, we have $G$ is a $C_\pi$-group and, since $G_{\mathcal{F}} \leq H_{\mathcal{F}} \leq H$, $H/H_{\mathcal{F}}$ is soluble.

Step 2. If $G$ has an $\mathcal{F}$-injector, then an $\mathcal{F}$-injector of $G$ is also an $\mathcal{F}$-injector of some Hall $\pi$-subgroup of $G$.

Let $V$ be an $\mathcal{F}$-injector of $G$. Assume that $V$ is a $\pi$-group of $G$. Without loss of generality, assume that $V \leq H$. Now prove that $V$ is also an $\mathcal{F}$-injector of $H$. 

17
Denote \( N = G \). Then set 
\[
\mathcal{F}^* = \{ M/N : M \in \mathcal{F}, N \leq M \}
\]
is a Fitting set of the soluble group \( G/N \).
Moreover, by [13, VIII-2.17(a)], have that 
\[
\mathcal{F}_0 = \{ S \leq G : SN/N \in \mathcal{F}^* \text{ and } S \text{ is subnormal in } SN \}
\]
is a Fitting set of \( G \). Observe that \( \mathcal{F}_0 \subset \mathcal{F} \) and, for any subnormal subgroup \( S \) of \( G \), \( S_{\mathcal{F}_0} = S \). By hypotheses of the theorem, \( G \) has a subnormal series
\[
1 \leq N = G \triangleleft G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n = G
\]
such that \( G_i/G_{i-1} \) is a \( G \)-chief factor and is an \( E^*_G \)-group. By [13, VIII-2.17(b)], if \( V/N \) is an \( \mathcal{F}^* \)-injector of \( H/N \), then \( V \) is a \( \mathcal{F}_0 \)-injector of \( H \). Since \( G \triangleleft H \), \( H \) has the subnormal series
\[
1 \leq G = H \cap G_0 \leq G_1 \cap H \leq G_2 \cap H \cdots \leq G_{n-1} \cap H \leq G_n \cap H = H
\]
such that \( G_i \cap H/G_{i-1} \cap H \) is an \( E^*_G \)-group by Lemma 2.6. And also \( V \) is an \( \mathcal{F} \)-injector of \( H \). To see that, we prove that, for any subnormal subgroup \( H_i = G_i \cap H \) of \( H \), the subgroup \( V \cap H_i \) is \( \mathcal{F} \)-maximal in \( H_i \). Suppose that there exists \( W \in \mathcal{F} \) such that \( V \cap H_i \leq W \leq H_i \). Then \( (V \cap H_i)N/N = (V/N) \cap (SN/N) \leq WN/N \leq H_iN/N \). Since \( H_i \mathcal{F}_0 = H_i \mathcal{F} \leq V \cap H_i \in \text{Inj}_{\mathcal{F}_0}(H_i) \), then \( H_i \mathcal{F} \leq W \). By Lemma 2.5, \( N \cap H_i = H_i \mathcal{F} \). Therefore \( W(N \cap H_i) = WH_i \mathcal{F} = W \), \( W \) is subnormal in \( WN \), and so \( WN \in \mathcal{F} \). Thus, \( WN/N \in \mathcal{F}^* \). Since \( (V/N) \cap (H_iN/N) \) is \( \mathcal{F}^* \)-maximal in \( H_iN/N \), \( (V \cap H_i)N = WN \), This means that \( V \cap H_i = (V \cap H_i)(N \cap H_i) = WN \cap H_i = W \), and \( V \cap H_i \) is \( \mathcal{F} \)-maximal in \( H_i \). Therefore, have that \( V \in \text{Inj}(G) \).

**Step 3.** If \( G \) have \( \mathcal{F} \)-injectors, then any two \( \mathcal{F} \)-injectors are conjugate in \( G \).

By [13, VIII-2.15], if \( V \in \text{Inj}(G) \), then \( V/N \) is an \( \mathcal{F}^* \)-injector of the soluble group \( G/N \). By Lemma 2.1, the \( \mathcal{F}^* \)-injectors of \( G/N \) are conjugate in \( G/N \). And so any two \( \mathcal{F} \)-injectors are conjugate in \( G \).

**Step 4.** \( G \) has an \( \mathcal{F} \)-injector.

Let \( H \) be a Hall \( \pi \)-subgroup of \( G \), Then \( H/H \) is soluble by step 1, and hence \( H \) has an \( \mathcal{F} \)-injector by [12, Theorem 2.4.27]. In order to prove that \( G \) has an \( \mathcal{F} \)-injector, only needs to prove an arbitrary \( \mathcal{F} \)-injector of \( H \) is an \( \mathcal{F} \)-injector of \( G \).

Let \( V \) be an \( \mathcal{F} \)-injector of \( H \). Let \( K \) be an subnormal subgroup of \( G \). Then \( V \cap K \) is a subnormal subgroup of \( H \). By Lemma 2.2(4), the subgroup \( V \cap K = (V \cap H) \cap K = V \cap (H \cap K) \) is an \( \mathcal{F} \)-injector of \( H \cap K \). Since \( |K : H \cap K| = |KH : H| \) is a \( \pi \)-number, \( H \cap K \) is a Hall \( \pi \)-subgroup of \( K \). So we need to deal with the following cases: \( K = G \) or \( K < G \).

**Case 1:** \( K < G \). Then by induction, \( V \cap K \) is an \( \mathcal{F} \)-injector of \( K \), and \( V \cap K \) is an \( \mathcal{F} \)-maximal subgroup of \( K \). Since \( K \) is arbitrary, \( V \) is also an \( \mathcal{F} \)-injector of \( G \).

**Case 2:** \( K = G \). Let \( W \) be an maximal \( \mathcal{F} \)-subgroup of \( G \) with \( V \leq W \leq G \). Since for every subnormal subgroup \( M \) of \( G \), \( W \cap M = V \cap M \) is an \( \mathcal{F} \)-maximal subgroup of \( M \).
by case 1. And so \( W \) is an \( \mathcal{F} \)-injector of \( G \). Since \( G \in C^s_\pi \), there exists an element \( x \in G \) such that \( V \leq W \leq H^x \). But, by Lemma 2.2(3), \( V^x \) is also an \( \mathcal{F} \)-injector of \( H^x \). By step 2, \( W \) is also an \( \mathcal{F} \)-injector of \( H \). Since \( H/H^x \) is soluble, by [12, Theorem 2.4.27], \( W \) and \( V \) are conjugate in \( G \), and so \( V = W \).

This completes the proof. \( \square \)

**Remark 3.1.** This Theorem 1.1 is comparing the Theorem 2.4.27 of [12].

**Corollary 3.1.** Let \( \mathcal{F} \) be a non-empty Fitting class and \( \pi = \pi(\mathcal{F}) \). If every chief factor for every maximal subgroup of \( G \) is an \( C^s_\pi \)-group, then \( G \) has an \( \mathcal{F} \)-injector.

**Proof.** 1. Let \( M_1, M_2 \) be maximal subgroups of \( G \) such that \( M_1, M_2 \) are not conjugate in \( G \). Then \( G = M_1M_2 \). By Theorem 3.1, \( M_1, M_2 \) have \( \mathcal{F} \)-injectors \( V_1, V_2 \).

If \( M_1 \cap M_2 = 1 \), then \( G = M_1 \times M_2 \), and, by [17, Lemma 1], \( G \) contains \( \mathcal{F} \)-injectors which are the product of the \( \mathcal{F} \)-injectors of the factors, \( M_1, M_2 \).

If \( M_1 \cap M_2 \neq 1 \), so there exists a prime \( p \) dividing the order of \( M_1 \cap M_2 \). And so assume that \( |G : M_1| = p \) or \( q \), where \( p \neq q \).

**Case 1:** If \( |G : M_1| = q \), then \( M_1 \triangleleft G \), and \( V_1 \), which is an \( \mathcal{F} \)-injector of \( M_1 \), is also an \( \mathcal{F} \)-injector of \( G \). To see this. Only needs to prove every subnormal subgroup \( K \) of \( G \), \( V_1 \cap K \) is an \( \mathcal{F} \)-injector of \( K \). By Lemma 2.5, \( M_1 \mathcal{F} = G \mathcal{F} \cap M_1 \). By hypotheses, there exists a series

\[
1 \leq W_0 = M_1 \mathcal{F} = G \mathcal{F} \cap M_1 \leq W_1 \leq W_2 \leq \cdots \leq W_{n-1} = M_1 \leq W_n = G
\]

such that every chief factor of \( G \) is \( E^s_\pi \)-group, then by Theorem 1.1, \( V_1 \) is an \( \mathcal{F} \)-injector of \( G \).

**Case 2:** If \( |G : M_1| = p \), then, for a Sylow \( p \)-subgroup \( P_1 \) of \( M_1 \), there exists a \( p \)-subgroup \( P_2 \) such that \( P = P_1P_2 \) is a Sylow \( p \)-subgroup of \( G \) and \( |P : P_1| = p \). If \( p \notin \pi(\mathcal{F}) \), by case 1, \( V_1 \) is an \( \mathcal{F} \)-injector of \( G \). If \( p \in \pi(\mathcal{F}) \), then there exists a Hall subgroup \( H \) such that \( H/H^x \) is soluble. So by Theorem 1.1, \( G \) has an \( \mathcal{F} \)-injector.

2. Let \( M_1, M_2 \) be maximal subgroups of \( G \) such that \( M_1, M_2 \) are conjugate in \( G \). Then \( M_1M_2 = M_2^gM_2 = M_2 \leq G \), for some \( g \in G \). Then, if \( M_2 < G \), by case 2, \( G \) also has an \( \mathcal{F} \)-injector. If \( M_2 = G \), by Lemma 2.6, and Theorem 1.1, \( G \) has an \( \mathcal{F} \)-injector.

This completes the proof. \( \square \)

**Remark 3.2.** If the condition of Corollary 3.1 is that, every chief factor is an \( E^m_\pi \), we also can get the same result.

**Corollary 3.2.** Let \( \mathcal{F} \) be a non-empty Fitting class. If every chief factor of \( G \) is an \( E^s_\pi \)-group, then \( A \) is an \( \mathcal{F} \)-injector of \( G \) if and only if \( A \) is a maximal \( \mathcal{F} \)-subgroup of \( G \) containing \( G \mathcal{F} \).
Acknowledgment.

The authors would like to thank the referee for the valuable suggestions and comments. This object is partially supported by Scientific Research Fund of School of Science of SUSE.

References


