Some fixed point theorems for pointwise R-weakly commuting hybrid mappings in metrically convex spaces

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Abstract

In the present paper we prove some coincidence common fixed point theorems for a family of hybrid pairs of mappings in metrically convex spaces by using the notion of pointwise R-weakly commuting mappings.

Key Words: Fixed point, hybrid contractive condition, metrically convex metric spaces, R-weakly commuting mappings.

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1 Introduction

Fixed point theorems for single-valued and multivalued mappings have been studied extensively and applied to diverse problems during the last few decades. Nadler [17] introduced the concept of multivalued contraction mappings and established that a multivalued contraction mapping possesses a fixed point in a complete metric space. Subsequently, many authors have generalized Nadler’s fixed point theorem in different ways. Assad and Kirk [4] gave sufficient conditions for non-self mappings to ensure the fixed point by proving a result on multivalued contractions in complete metrically convex metric spaces. Several authors proved some fixed point theorems for non-self mappings (see, for instance [1], [2], [11], [12], [13], [15], [19]).

Recently, Imdad and Khan [12] and Dhage, Dolhare and Petrusel [8] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due
to Itoh [13], Khan [15], Ahmad and Imdad [1, 2], Ahmad and Khan [3] and others. The purpose of this paper is to prove some coincidence and common fixed point theorems for a sequence of hybrid type non-self mappings satisfying certain contraction condition by using R-weakly commutativity between multivalued mappings and single-valued mappings. Our results generalize and unify the results due to Imdad and Khan [12], Khan [15], Itoh [13], Ahmad and Imdad [1, 2], Ahmad and Khan [3] and several others.

2 Preliminaries

Let (X, d) be a metric space. Then following Nadler [17], we recall

(i) \( CB(X) = \{ A: A \) is nonempty closed and bounded subset of X\}

(ii) \( C(X) = \{ A: A \) is nonempty compact subset of X\}

(iii) For nonempty subsets \( A, B \) of \( X \) and \( x \in X, d(x, A) = \inf\{d(x, a) : a \in A\}, \)

\[
H(A, B) = \max[\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}]. \tag{1}
\]

It is well known that \( CB(X) \) is a metric space with the distance \( H \) which is known as Hausdroff-Pompeiu metric on \( X \).

The following definitions and lemmas will be frequently used in the sequel.

**Definition 1** [10]. Let \( K \) be a nonempty subset of a metric space \( (X, d) \), \( T : K \to X \) and \( F : K \to CB(X) \). The pair \((F, T)\) is said to be pointwise R-weakly commuting on \( K \) if for given \( x \in K \) and \( Tx \in K \), there exists some \( R = R(x) > 0 \) such that

\[
d(Ty, FTx) \leq R.d(Tx, Fx) \tag{2}
\]

for each \( y \in K \cap Fx \). Moreover, the pair \((F, T)\) will be called R-weakly commuting on \( K \) if \( (2) \) holds for each \( x \in K \) and \( Tx \in K \) with some \( R > 0 \).

If \( R = 1 \), we get the definition of weak commutativity of \((F, T)\) on \( K \). For \( K = X \) definition [10] reduces to “Pointwise R-weakly commutativity” for single valued self mappings due to Pant [18].

**Definition 2** [9, 10]. Let \( K \) be a nonempty subset of a metric space \( (X, d) \), \( T : K \to X \) and \( F : K \to CB(X) \). The pair \((F, T)\) is said to be weakly commuting if for every \( x, y \in K \) with \( x \in Fy \) and \( Ty \in K \), we have

\[
d(Tx, FTy) = d(Ty, Fy). \tag{3}
\]
Definition 3 [10]. Let $K$ be a nonempty subset of a metric space $(X, d)$, $T : K \to X$ and $F : K \to CB(X)$. The pair $(F, T)$ is said to be compatible if for every sequence $\{x_n\} \subset K$, from the relation
\[
\lim_{n \to \infty} d(Fx_n, Tx_n) = 0 \tag{4}
\]
and $Tx_n \in K$ (for every $n \in \mathbb{N}$) it follows that $\lim_{n \to \infty} d(Ty_n, FTx_n) = 0$, for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n$, $n \in \mathbb{N}$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Seesa [14].

Definition 4 [11]. Let $K$ be a nonempty subset of a metric space $(X, d)$, $T : K \to X$ and $F : K \to CB(X)$. The pair $(F, T)$ is said to be quasi-coincidentally commuting if for all coincidence points ‘$x$’ of $(T, F)$, $TFx \subset FTx$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.

Definition 5 [11]. A mapping $T : K \to X$ is said to be coincidentally idempotent w.r.t. mapping $F : K \to CB(X)$, if $T$ is idempotent at the coincidence points of the pair $(F, T)$.

Definition 6 [4]. A metric space $(X, d)$ is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X$, $x \neq z \neq y$ such that
\[
d(x, z) + d(z, y) = d(x, y). \tag{5}\]

Lemma 1 [4]. Let $K$ be a nonempty closed subset of a metrically convex metric space $(X, d)$, if $x \in K$ and $y \notin K$ then there exists a point $z \in \partial K$ (the boundary of $K$) such that
\[
d(x, z) + d(z, y) = d(x, y).\]

Lemma 2 [17]. Let $A, B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$ there exists $b = b(a)$ in $B$ such that $q.d(a, b) = H(A, B)$.

3 Main results

Theorem 1 Let $(X, d)$ be a complete metrically convex metric space and $K$ is a nonempty closed subset of $X$. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ satisfying

\begin{enumerate}[(iv)]
\item $\delta K \subseteq SK \cap TK$, $F_i(K) \cap K \subseteq SK$, $F_j(K) \cap K \subseteq TK$ \tag{6}
\item $Tx \in \delta K \Rightarrow F_i(x) \subseteq K$, $Sx \in \delta K \Rightarrow F_j(x) \subseteq K$ and
\[
H[F_i(x), F_j(y)] \leq ad(Tx, Sy) + b \max\{d(Tx, F_i(x)), d(Sy, F_j(y))\}
\]
+ $c \max\{d(Tx, F_i(x)), d(Tx, F_i(x)), d(Sy, F_j(y))\} \tag{6}$
\end{enumerate}

where $i = 2n - 1$, $j = 2n$, $(n \in \mathbb{N})$, $i \neq j$ for all $x, y \in K$ with $x \neq y, a, b \geq 0$ and
\[
\{(a + 2b + 2c) + (a^2 + ab + ac)/q\} < q < 1,
\]

137
Proof. Firstly, we proceed to construct two sequences \( \{x_n\} \) and \( \{y_n\} \) in the following way:

Let \( x \in \delta K \). Since \( \delta K \subseteq TK \) there exists a point \( x_0 \in K \) such that \( x = Tx_0 \). From the implication \( Tx_0 \in \delta K \) which implies \( F_1(x_0) \subseteq F_1(K) \cap K \subseteq SK \). Let \( x_1 \in K \) be such that \( y_1 = Sx_1 \in F_1(x_0) \subseteq K \). Since \( y_1 \in F_1(x_0) \) there exists a point \( y_2 \in F_2(x_1) \) such that

\[
q.d(y_1, y_2) \leq H[F_1(x_0), F_2(x_1)]
\]

Suppose \( y_2 \in K \). Then \( y_2 \in F_2(K) \cap K \subseteq TK \) implies that there exists a point \( x_2 \in K \) such that \( y_2 \in Tx_2 \). Otherwise, if \( y_2 \not\in K \), then there exists a point \( p \in \delta K \) such that

\[
d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2)
\]

Since \( p \in \delta K \subseteq TK \), there exists a point \( x_2 \in K \) with \( p = Tx_2 \) so that

\[
d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2)
\]

Let \( y_3 \in F_3(x_2) \) be such that

\[
q.d(y_2, y_3) \leq H[F_2(x_1), F_3(x_2)]
\]

Thus on repeating the foregoing arguments, we obtain two sequences \( \{x_n\} \) and \( \{y_n\} \) such that

(iii) \( y_{2n} \in F_{2n}(x_{2n-1}), y_{2n+1} \in F_{2n+1}(x_{2n}) \),

(ix) \( y_{2n} \in K \Rightarrow y_{2n} =Tx_{2n} \) or \( y_{2n} \not\in K \Rightarrow Tx_{2n} \in \delta K \) and

\[
d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})
\]

(x) \( y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1} \) or \( y_{2n+1} \not\in K \Rightarrow Sx_{2n+1} \in \delta K \) and

\[
d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})
\]

We denote

\[
\begin{align*}
P_0 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}, \} \\
P_1 &= \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}, \} \\
Q_0 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}, \} \\
Q_1 &= \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}, \}
\end{align*}
\]
First we show that \((Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1\) and \((Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1\). If \(Tx_{2n} \in P_1\), then \(y_{2n} \neq Tx_{2n}\) and we have \(Tx_{2n} \in \delta K\) which implies that \(y_{2n+1} \in F_{2n+1}(x_{2n}) \subseteq K\). Hence \(y_{2n+1} = Sx_{2n+1} \in Q_0\). Similarly, one can argue that \((Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1\).

Now we distinguish the following three cases:

**Case 1.** If \((Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_0\), then

\[
q.d(Tx_{2n}, Sx_{2n+1}) \leq H[F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})]
\]

\[
\leq ad(Tx_{2n}, Sx_{2n-1}) + b \max\{d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\}
\]

\[
+ c \max\{d(Tx_{2n}, Sx_{2n+1}), d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Tx_{2n}, F_{2n+1}(x_{2n}))\}
\]

\[
\leq ad(y_{2n}, y_{2n-1}) + b \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}
\]

\[
+ c \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}
\]

which in turn yields

\[
d(Tx_{2n}, Sx_{2n+1}) \leq \left\{ \begin{array}{l}
\left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n}), \text{ if } d(y_{2n-1}, y_{2n}) \geq d(y_{2n+1}, y_{2n}) \\
\left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n}), \text{ if } d(y_{2n-1}, y_{2n}) \leq d(y_{2n+1}, y_{2n})
\end{array} \right.
\]

or

\[
d(Tx_{2n}, Sx_{2n+1}) \leq hd(Sx_{2n-1}, Tx_{2n}),
\]

where \(h = \max\{(a+b+c)/q, (a/(q-b-c))\} < 1\), since \{(a+2b+2c)+(a^2+ab+ac)/q\} < 1.

Similarly if \((Sx_{2n-1}, Tx_{2n}) \in Q_0 \times P_0\), then

\[
d(Sx_{2n-1}, Tx_{2n}) \leq \left\{ \begin{array}{l}
\left(\frac{a+b+c}{q}\right) d(Sx_{2n-1}, Tx_{2n}), \text{ if } d(y_{2n-2}, y_{2n-1}) \geq d(y_{2n-1}, y_{2n}) \\
\left(\frac{a}{q-b-c}\right) d(Sx_{2n-1}, Tx_{2n}), \text{ if } d(y_{2n-2}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n})
\end{array} \right.
\]

or

\[
d(Sx_{2n-1}, Tx_{2n}) \leq h.d(Sx_{2n-1}, Tx_{2n-2}),
\]

where \(h = \max\{(a+b+c)/q, (a/(q-b-c))\} < 1\), since \{(a+2b+2c)+(a^2+ab+ac)/q\} < 1.

**Case 2.** If \((Tx_{2n}, Sx_{2n+1}) \in P_0 \times Q_1\), then

\[
d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})
\]

which in turn yields

\[
d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1})
\]

and hence

\[
q.d(Tx_{2n}, Sx_{2n+1}) \leq q.d(y_{2n}, y_{2n+1})
\]

\[
\leq H[F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})].
\]

139
Now proceeding as in case 1, we have

\[ d(Tx_{2n}, Sx_{2n+1}) \leq \begin{cases} \left\{ \frac{a+b+c}{q} \right\} d(Sx_{2n-1}, Tx_{2n}), & \text{if } d(y_{2n-1}, y_{2n}) \geq d(y_{2n+1}, y_{2n}) \\ \left\{ \frac{a}{q-b-c} \right\} d(Sx_{2n-1}, Tx_{2n}), & \text{if } d(y_{2n-1}, y_{2n}) \leq d(y_{2n+1}, y_{2n}), \end{cases} \]

or

\[ d(Tx_{2n}, Sx_{2n+1}) \leq h d(Sx_{2n-1}, Tx_{2n}), \]

where \( h = \max \left\{ \frac{(a + b + c)/q}{q}, \frac{(a/(q - b - c))}{q} \right\} < 1 \), since \( \{(a + 2b + 2c) + (a^2 + ab + ac)/q \} < 1 \). Similarly if \( (Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_0 \), then

\[ d(Sx_{2n-1}, Tx_{2n}) \leq \begin{cases} \left\{ \frac{a+b+c}{q} \right\} d(Sx_{2n-1}, Tx_{2n-2}), & \text{if } d(y_{2n-2}, y_{2n-1}) \geq d(y_{2n-1}, y_{2n}) \\ \left\{ \frac{a}{q-b-c} \right\} d(Sx_{2n-1}, Tx_{2n-2}), & \text{if } d(y_{2n-2}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n}), \end{cases} \]

or

\[ d(Sx_{2n-1}, Tx_{2n}) \leq h \cdot d(Sx_{2n-1}, Tx_{2n-2}), \]

where \( h = \max \left\{ \frac{(a + b + c)/q}{q}, \frac{(a/(q - b - c))}{q} \right\} < 1 \).

**Case 3.** If \( (Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_0 \), then \( Sx_{2n-1} = y_{2n-1} \). Now proceeding as in case 1, one gets

\[ q \cdot d(Tx_{2n}, Sx_{2n+1}) = q \cdot d(Tx_{2n}, y_{2n+1}) \leq q \cdot d(Tx_{2n}, y_{2n}) + q \cdot d(y_{2n}, y_{2n+1}) \]

\[ \leq q \cdot d(Sx_{2n-1}, y_{2n}) + H[F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})] \]

\[ \leq q \cdot d(Sx_{2n-1}, y_{2n}) + ad(y_{2n}, y_{2n-1}) + b \cdot \max \{d(y_{2n}, y_{2n-1}), d(y_{2n-1}, y_{2n})\} \]

\[ + c \cdot \max \{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1})\}, \]

which in turn yields

\[ d(Tx_{2n}, Sx_{2n+1}) \leq \begin{cases} \left\{ \frac{a+b+c}{q} \right\} d(Sx_{2n-1}, Tx_{2n}), & \text{if } d(y_{2n-1}, y_{2n}) \geq d(y_{2n+1}, y_{2n}) \\ \left\{ \frac{a}{q-b-c} \right\} d(Sx_{2n-1}, Tx_{2n}), & \text{if } d(y_{2n-1}, y_{2n}) \leq d(y_{2n+1}, y_{2n}). \end{cases} \]

Now proceeding as earlier, one also obtain

\[ d(Sx_{2n-1}, Tx_{2n}) \leq \begin{cases} \left\{ \frac{a+b+c}{q} \right\} d(Sx_{2n-1}, Tx_{2n-2}), & \text{if } d(y_{2n-2}, y_{2n-1}) \geq d(y_{2n-1}, y_{2n}) \\ \left\{ \frac{a}{q-b-c} \right\} d(Sx_{2n-1}, Tx_{2n-2}), & \text{if } d(y_{2n-2}, y_{2n-1}) \leq d(y_{2n-1}, y_{2n}). \end{cases} \]

Therefore combining above inequalities, we have

\[ d(Tx_{2n}, Sx_{2n+1}) \leq k \cdot d(Sx_{2n-1}, Tx_{2n-2}) \]

where

\[ k = \max \left\{ \left\{ \frac{a+b+c}{q} \right\}, \left\{ \frac{q+a}{q-b-c} \right\}, \left\{ \frac{a+b+c}{q} \right\}, \left\{ \frac{q+a+b+c}{q} \right\}, \left\{ \frac{a}{q-b-c} \right\}, \left\{ \frac{q+a+b+c}{q} \right\}, \left\{ \frac{a}{q-b-c} \right\}, \left\{ \frac{q+a+b+c}{q} \right\} \right\} < 1, \]
since \( \{(a + 2b + 2c) + (a^2 + ab + ac)/q\} < 1 \).

To substantiate that, the inequality \( \{(a + 2b + 2c) + (a^2 + ab + ac)/q\} < 1 \) implies all foregoing inequalities, one may note that
\[
\{(a + 2b + 2c) + (a^2 + ab + ac)/q\} < q \Rightarrow \{(aq + 2bq + 2cq) + (a^2 + ab + ac)\} < q^2,
\]
\[
aq + a^2 + bq + ab + cq + ac + bq + cq < q^2,
\]
or
\[
aq + a^2 + bq + ab + cq + ac < q^2 - bq - cq,
\]
or
\[
\left(\frac{a + b + c}{q}\right) \left(\frac{q + a}{q - b - c}\right) < 1
\]
and
\[
\{(a + 2b + 2c) + (a^2 + ab + ac)/q\} < q \Rightarrow \{(a + b + c) + (a^2 + ab + ac)/q\} < q
\]
or
\[
\{(aq + bq + cq) + (a^2 + ab + ac)\} < q^2,
\]
or
\[
aq + a^2 + ab + ac + bq + cq < q^2,
\]
or
\[
aq + a^2 + ab + ac < q^2 - bq - cq
\]
or
\[
\left(\frac{a}{q - b - c}\right) \left(\frac{q + a + b + c}{q - a}\right) < 1.
\]

Similarly one can establish the other inequalities as well. Thus in all the cases we have
\[
d(Tx_{2n}, Sx_{2n+1}) \leq k \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}
\]
whereas
\[
d(Sx_{2n+1}, Tx_{2n+1}) \leq k \max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}
\]
Now on the lines of Assad and Kirk [4], it can be shown by induction that for \( n = 1 \), we have
\[
d(Tx_{2n}, Sx_{2n+1}) \leq k^n \delta, d(Sx_{2n+1}, Tx_{2n+2}) \leq k^{n+\frac{1}{2}} \delta
\]
Whereas
\[
\delta = k^{-\frac{1}{2}} \max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}
\]
Thus the sequence \( \{Tx_0, Sx_1, Tx_2, Sx_3, ... Tx_{2n}, Sx_{2n+1}\} \) is a Cauchy sequence and hence converges to a point \( z \) in \( X \). Now we assume that there exists a subsequence \( \{Tx_{2n_k}\} \) of \( \{Tx_{2n}\} \) which is contained in \( P_0 \). Further subsequences \( \{Tx_{2n_k}\} \) and \( \{Sx_{2n_k+1}\} \) both converge to \( z \in K \) as \( K \) is closed subset of the complete metric space \( (X, d) \). Since \( Tx_{2n_k} \in F_j(x_{2n_k-1}) \)
for any even integer \( j \in N \) and \( Sx_{2n_k-1} \in K \). Using pointwise R-weak commutativity of \((F_j, S)\), we have
\[
\text{d}(SF_j(x_{2n_k-1})), F_j(Sx_{2n_k-1})) \leq R_1 \text{d}(F_j(x_{2n_k-1}), Sx_{2n_k-1}))
\]  
(8)
for every even integer \( j \in N \) with some \( R_1 > 0 \). Also
\[
\text{d}(SF_j(x_{2n_k-1})), F_j(z)) \leq \text{d}(SF_j(x_{2n_k-1})), F_j(Sx_{2n_k-1}))+H(F_j(x_{2n_k-1})), F_j(z)).
\]  
(9)
Making \( k \to \infty \) in (8) and (9) and using the continuity of \( S \) and \( F_j \), we get \( \text{d}(Sz,F_j(z)) \leq 0 \)
yielding thereby \( Sz \in F_j(z) \), for any even integer \( j \in N \).
Since \( y_{2n_k+1} \in F_i(x_{2n_k}) \) and \( Tx_{2n_k} \in K \) for any odd integer \( i \in N \). Using pointwise R-weak commutativity of \((F_i, T)\), we have
\[
\text{d}(TF_i(x_{2n_k})), F_i(Tx_{2n_k}) \leq R_2 \text{d}(F_i(x_{2n_k})), Tx_{2n_k})
\]  
for every odd integer \( i \in N \) with some \( R_2 > 0 \), besides
\[
\text{d}(TF_i(x_{2n_k})), F_i(z)) \leq \text{d}(TF_i(x_{2n_k})), F_i(Tx_{2n_k}))+H(F_i(x_{2n_k})), F_i(z)).
\]
Therefore as earlier the continuity of \( F_i \) and \( T \) implies \( \text{d}(Tz,F_i(z)) \leq 0 \)
yielding thereby \( Tz \in F_i(z) \), for any odd integer \( i \in N \) as \( k \to \infty \).
If we assume that there exists a subsequence \( \{Sx_{2n_k+1}\} \) contained in \( Q_0 \), then analogous arguments establish the earlier conclusions. This concludes the proof.

**Remark 1** If we replace condition (6) by the condition
\[
H[F_i(x), F_j(y)] \leq a \max\{\frac{1}{2} \text{d}(Tx, Sy), \text{d}(Tx, F_i(x)), \text{d}(Sy, F_j(y))\}
\]
\[
+ b\{\text{d}(Tx, F_j(y)) + \text{d}(Sy, F_i(x))\}
\]
then we get Theorem 3.4 [12].

**Remark 2** If we replace condition (6) by the condition
\[
H[F_i(x), F_j(y)] \leq a \max\{\frac{1}{2} \text{d}(Tx, Sy), \text{d}(Tx, F_i(x)), \text{d}(Sy, F_j(y))\}
\]
\[
+ b\{\text{d}(Tx, F_j(y)) + \text{d}(Sy, F_i(x))\}
\]
and pointwise R-weakly commuting maps by compatible maps, then we get Theorem 3.1 due to Imdad and Khan [12].

**Theorem 2** Let \((X, d)\) be a complete metrically convex metric space and \( K \) is a nonempty closed subset of \( X \). Let \( \{F_n\}_{n=1}^{\infty} : K \to CB(X) \) and \( S, T : K \to X \) satisfying (6), (iv) and (v). Suppose that
\((\xi)\) \( TK \) and \( SK \) are closed subspaces of \( X \). Then
(*) \((F_i, T)\) has a point of coincidence,

(**) \((F_j, S)\) has a point of coincidence.

Moreover, \((F_i, T)\) has a common fixed point if \(T\) is quasi-coincidentally commuting and coincidentally idempotent w.r.t. \(F_i\) whereas \((F_j, S)\) has a common fixed point provided \(S\) is quasi-coincidentally commuting and coincidentally idempotent w.r.t. \(F_j\).

**Proof.** On the lines of Theorem 1, one assumes that there exists a subsequence \(\{Tx_{2n_k}\}\) which is contained in \(P_0\) and \(TK\) as well as \(SK\) are closed subspaces of \(X\). Since \(\{Tx_{2n_k}\}\) is Cauchy in \(TK\), it converges to a point \(u \in TK\). Let \(v \in T^{-1}u\), then \(Tv = u\). Since \(\{Sx_{2n_k+1}\}\) is a subsequence of Cauchy sequence, \(\{Sx_{2n_k+1}\}\) converges to \(u\) as well. Using (6), one can write

\[
q.d(F_i(v), Tx_{2n_k}) \leq H[F_i(v), F_j(x_{2n_k-1})]
\leq ad(Tv, Sx_{2n_k-1}) + b \max\{d(Tv, F_i(v)), d(Sx_{2n_k-1}, F_j(x_{2n_k-1}))\}
+ c \max\{d(Tv, Sx_{2n_k-1}), d(Tv, F_i(v)), d(Sx_{2n_k-1}, F_j(x_{2n_k-1}))\}
\]

which on letting \(k \to \infty\), reduces to

\[
q.d(F_i(v), u) \leq a(0) + b \max\{d(u, F_i(v)), 0\} + c \max\{0, d(u, F_i(v)), 0\}
\leq (b + c).d(u, F_i(v)),
\]

yielding thereby \(u \in F_i(v)\) which implies that \(u = Tv \in F_i(v)\) as \(F_i(v)\) is closed.

Since Cauchy sequence \(\{Tx_{2n}\}\) converges to \(u \in K\) and \(u \in F_i(v), u \in F_i(K) \cap K \subseteq SK\), there exists \(w \in K\) such that \(Sw = u\). Again using (6), one gets

\[
q.d(Sw, F_j(w)) = q.d(Tv, F_j(w)) \leq H[F_i(v), F_j(x_{2n_k-1})]
\leq ad(Tv, Sw) + b \max\{d(Tv, F_i(v)), d(Sw, F_j(w))\}
+ c \max\{d(Tv, Sw), d(Tv, F_i(v)), d(Sw, F_j(w))\} \leq (b + c).d(Sw, F_j(w))
\]

implying thereby \(Sw \in F_j(w)\), that is \(w\) is a coincidence point of \((S, F_j)\).

If one assumes that there exists a subsequence \(\{Sx_{2n_k+1}\}\) contained in \(Q_0\) with TK as well as SK are closed subspaces of X, then noting that \(\{Sx_{2n_k+1}\}\) is Cauchy in SK, the foregoing arguments establish that \(Tv \in F_i(v)\) and \(Sw \in F_j(w)\).

Since \(v\) is a coincidence point of \((F_i, T)\) therefore using quasi-coincidentally commuting property of \((F_i, T)\) and coincidentally idempotent property of \(T\) w.r.t. \(F_i\), one can have

\[
Tv \in F_i(v), u = Tv \Rightarrow Tu = TTv = Tv = u,
\]

therefore \(u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)\) which shows that \(u\) is a common fixed point of \((F_i, T)\). Similarly using the quasi-coincidentally commuting property of \((F_j, S)\) and coincidentally idempotent property of \(S\) w.r.t. \(F_j\), one can show that \((F_j, S)\) has a common fixed point as well.
References


